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Mathematical Reviews

Vol. 20, No. 9

OCTOBER, 1959

Reviews 5724-6344

LOGIC AND FOUNDATIONS

See also 5989.

5724:

Piaget, Jean. *Méthode axiomatique et méthode opérationnelle.* *Synthese* 10, 23-43.

In the author's opinion the laws of logic express, in some manner, those of intelligence or of thinking; thus it is necessary, from the viewpoint of psychology (which aims at "explaining" rather than at providing a "foundation"), to set up, for each axiomatic system of logic, a "pre-axiomatic" analysis showing the operational structures which the axioms are meant to control. On the epistemological level, this conception calls for a psychogenetic analysis with the operational method as a first step towards the more complete formalisation provided by the axiomatic method. *E. W. Beth* (Amsterdam)

5725:

Kazemier, B. H. *Formal systems and mental activity.* *Synthese* 10, 210-219.

Nach einleitenden Bemerkungen über Sprache als eine Form menschlichen Verhaltens gibt Verf. eine kurze, aber klare Darstellung von Sinn und Aufgabe der Hilbertschen Metamathematik und der Tarskischen Semantik. Verf. ist bereit, die Beschäftigung mit beliebigen, auch uninterpretierten Formalismen gelten zu lassen. Behauptungen über Formalismen müssten jedoch einen konstruktiven Sinn haben. Die semantische "Interpretation" mengentheoretischer Formalismen mit Hilfe einer nicht-konstruktiven Metasprache ist daher für den Verf. keine Interpretation. *P. Lorenzen* (Kiel)

5726:

Shoenfield, J. R. *Open sentences and the induction axiom.* *J. Symb. Logic* 23 (1958), 7-12.

Peano arithmetic without multiplication is formalized so that induction appears as a rule of inference applicable only to open sentences (i.e., formulas with no bound variables).

The weakness of this induction rule is indicated by two main results. First, the induction rule can be replaced by four open sentences; namely, the commutative, associative and cancellation laws of addition, and a law for the predecessor. Secondly, the system containing the induction rule cannot be extended, through addition of finitely many open axioms, to include all true open sentences.

If, on the other hand, the above induction rule is replaced by a rule of double induction (the latter again applicable only to open sentences), the system now includes all true open sentences.

C. F. Rose (Santa Monica, Calif.)

5727:

Beth, E. W. *On machines which prove theorems.* *Simon Stevin* 32 (1958), 49-60.

This is a discussion of the problem of having theorems

proved by machines, the use of a decision method being restricted either arbitrarily, as in the case of the calculus of propositions, or because no such method exists, as in the lower predicate calculus. The author stresses the possible use of his own method of semantic tableaux in this connection. While the introduction of some heuristic devices may be useful, others, though important to a human agent, have, in the author's view, no place in a programme designed for a machine.

A. Robinson (Jerusalem)

5728:

Beth, E. W. *Remarks on elementary predicate logic.* *Nieuw Arch. Wisk.* (3) 5 (1957), 58-62.

This is a collection of a number of observations on the first order predicate calculus, semantic tableaux, and the analysis of intuitionistic predicate logic. In particular, the author stresses that a closed semantic tableau may be read as a formal derivation, and discusses the intuitionistic version of Herbrand's theorem.

A. Robinson (Jerusalem)

5729:

Beth, E. W. *On the completeness of the classical sentential logic.* *Nederl. Akad. Wetensch. Proc. Ser. A* 61=Indag. Math. 20 (1958), 434-437.

Verf. gibt mit Hilfe seiner Methode der semantischen Tableaus, die er aus dem Gentzenschen Sequenzen-Kalkül entwickelt hat, einen kurzen Beweis der Vollständigkeit der klassischen Aussagenlogik für Negation und die sog. materiale Implikation.

P. Lorenzen (Kiel)

5730:

Fitch, Frederic B. *A definition of existence in terms of abstraction and disjunction.* *J. Symb. Logic* 22 (1957), 343-344.

It is shown that, within the system *K* of basic logic [same *J.* 18 (1953), 317-325; *MR* 15, 924], the existence operator '*E*' may be suitably defined in terms of two-place abstraction and disjunction, with the requisite property that '*Ex*' is in *K* if and only if there exists an '*e*' such that '*be*' is in *K*.

R. M. Martin (Philadelphia, Pa.)

5731:

Fraenkel, Abraham A. *Paul Bernays und die Begründung der Mengenlehre.* *Dialectica* 12 (1958), 274-279.

5732:

Péter, Rózsa. *The boundedly recursive functions of Grzegorzczuk and the majorisation method of Ackermann.* *Mat. Lapok* 8 (1957), 93-99. (Hungarian)

5733:

Dekker, J. C. E.; and Myhill, J. *Retraceable sets.* *Canad. J. Math.* 10 (1958), 357-373.

Roughly speaking, a retraceable set α of non-negative integers is one for which there exists an effective procedure which when applied to any member of α other than its minimum yields the next smaller member of α . There are just c non-recursive sets which are retraceable, where c

is the cardinality of the continuum. In this paper several theorems concerning retraceable sets are proven; in particular, that every degree of unsolvability can be represented by a retraceable set, and that every degree of unsolvability which can be represented by a recursively enumerable set can also be represented by a recursively enumerable set with a retraceable complement.

R. M. Martin (Philadelphia, Pa.)

5734:

Smullyan, Raymond M. **Undecidability and recursive inseparability.** *Z. Math. Logik Grundlagen Math.* 4 (1958), 143-147.

The author proves two results on undecidability in arithmetic. (i) If every recursive number set S is representable in a theory T , i.e., if S coincides with the set of numbers for which a certain one place predicate is provable, then T is undecidable. (ii) If every recursive set is definable in T , T consistent, then the sets of true and false sentences of T are recursively inseparable. (i) represents an improvement upon a result of Myhill [*Z. Math. Logik Grundlagen Math.* 1 (1955), 97-108; MR 17, 118]. As stated by the author, the possibility of such an improvement had been pointed out earlier by P. Bernays.

A. Robinson (Jerusalem)

5735:

Kolmogorov, A. N.; and Uspenskii, V. A. **On the definition of an algorithm.** *Uspehi Mat. Nauk* 13 (1958), no. 4 (82), 3-28. (Russian)

A definition of algorithm is proposed which seems to be more general than the usual definitions in that it is not restricted to "words" or integers. An algorithm is given in the following way. Let T_0, T_1, \dots, T_k be pairwise-disjoint sets, and let $T = T_0 \cup T_1 \cup \dots \cup T_k$. A "state" S is a certain kind of finite one-dimensional complex with vertices in T . Sets of "initial" and "terminal" states are singled out, and rules for transforming a given state into some other state are provided. The algorithm is "applicable" to a given state if, beginning with this state, a finite sequence of transformations leads to a terminal state; from the terminal state, a certain sub-complex called the "solution" is derived. A standard way of representing n -tuples of natural numbers as states is given, and the authors claim that every partial-recursive function is computable by an algorithm. Conversely, by means of an appropriate assignment of "Gödel-numbers" to the states, each algorithm corresponds to a partial-recursive function. As an example, the authors give an algorithm for the function $2n$.

E. Mendelsohn (New York, N.Y.)

5736:

*Touchais, M. **Les applications techniques de la logique.** Préface de G. Lehmann. Dunod, Paris, 1956. xix+82 pp. 940 francs.

Introduction: le schéma électrique et la logique. Première partie: le calcul logique ou théorie de l'alternative: I: les opérations binaires; II: réduction du nombre des signes; III: fonctions d'un nombre quelconque de variables; IV: calcul des signes. Deuxième partie: applications: I: calcul des schémas de commutation; II: exemples d'applications et exercices; III: verrouillages et enclenchements; IV: applications diverses.

Table des matières

5737:

*Freudenthal, H. **Logique mathématique appliquée.** Collection de Logique Mathématique. Série A. XIV. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain; 1958. 58 pp. 1200 francs.

The contents of this book (which was awarded the prize in the Essay Contest "Mathematical logic as a tool of analysis" by The Institute for the Unity of Science in 1953) do not exactly correspond to its title. The only application described in detail (and very clearly indeed) is that of the sentential calculus in the theory and design of digital computers. The author also discusses the influence of mathematical logic on philosophy, and in this context he criticises various current conceptions concerning, in particular, the application of quotation marks in connection with the distinction between "use" and "mention" of a symbol, the interpretation of variables, the concept of implication, and the intensional and modal elements in discourse. Although on many of these topics the reviewer does not agree with the author, he considers his observations incisive and challenging.

E. W. Beth (Amsterdam)

5738:

Specker, Ernst. **Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom).** *Z. Math. Logik Grundlagen Math.* 3 (1957), 173-210.

The system of set-theory under discussion is that formulated by P. Bernays, and the numbering of axioms is taken from his series of articles [*J. Symb. Logic* 2 (1937), 65-77; 6 (1941), 1-17; 7 (1942), 65-89, 133-145; 8 (1943), 89-106; 13 (1948), 65-79; MR 2, 210; 3, 290; 4, 183; 5, 198; 10, 3]. There are three sections. The first contains a proof of the independence of the Fundierungsaxiom VII from the axioms of set-theory I, II, III, Va, the Paarklassenaxiom and VI. To construct a model which establishes this independence, the author shows that in certain special partially ordered systems it is possible to define a membership-relation \in_M which will satisfy the axioms of set-theory and also " $(\exists x)(x \in_M x)$ ". The second section is devoted to a proof of the independence of the axiom of choice from axioms I, II, III and V. The results overlap those of Shoenfield [*J. Symb. Logic* 20 (1956), 202] and Mendelson [*ibid.* 21 (1956), 350-366; MR 18, 864]. In their proofs the role of the Urelemente of Fraenkel and Mostowski is taken by infinite descending sequences, while in Specker's proof it is taken by elements which are their own unit classes. The model constructed also shows that a number of theorems are not provable without the axiom of choice and Fundierungsaxiom, e.g., that for cardinals $m \geq 5$, $m^2 < 2^m$. The third section concerns itself with the consistency of various alternatives to the axiom of choice relative to the axioms of set-theory and is related to the work of Church [*Trans. Amer. Math. Soc.* 29 (1927), 178-208].

L. N. Gál (New Haven, Conn.)

5739:

Rieger, Ladislav. **A contribution to Gödel's axiomatic set theory. I.** *Czechoslovak Math. J.* 7(82) (1957), 323-357. (Russian summary)

Let \mathcal{G} be a set-theory based on Gödel's axioms A, B, C, E and the generalized continuum hypothesis. The main result of this paper is that axiom C and the axiom of constructibility are independent of the axioms of \mathcal{G} , and that " $(\exists x)(x \in x)$ " is consistent with them. The proof involves the construction of a model of \mathcal{G} without using axiom D, using Robinson's definition of an ordinal [cf.

R. M. Robinson. *J. Symb. Logic* 2 (1937), 29-36] and also refers to the series of articles by P. Bernays [in particular, *ibid.* 13 (1948), 65-79; MR 10, 3; p. 67]. The reviewer was, however, unable to follow all of the details of the argument. For the independence of axiom D see also E. Mendelson, *ibid.* 21 (1956), 350-366 [MR 18, 864] and E. Specker [≠5738 above]. *L. N. Gdl* (New Haven, Conn.)

SET THEORY

See also 5746.

5740:

Bruns, Günter; und Schmidt, Jürgen. Eine Verschärfung des Bernsteinschen Äquivalenzsatzes. *Math. Ann.* 135 (1958), 257-262.

Let \mathcal{S} and \mathcal{T} be systems of subsets of E and F , respectively; the pairs (E, \mathcal{S}) and (F, \mathcal{T}) are called isomorphic if there exists a one-one mapping of E onto F that carries \mathcal{S} onto \mathcal{T} . Theorem: If (E, \mathcal{S}) is isomorphic to a trace of (F, \mathcal{T}) and (F, \mathcal{T}) is isomorphic to a trace of (E, \mathcal{S}) , and if \mathcal{S} and \mathcal{T} are convex lattices, then (E, \mathcal{S}) and (F, \mathcal{T}) are isomorphic to each other.

L. Gillman (Princeton, N.J.)

5741:

Čupona, G. On the relation distributivity between binary operations. *Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire* 9 (1956), 21-29. (Macedonian. English summary)

The paper is connected with the work of V. D. Belousov [Mat. Sb. N.S. 36 (78) (1955), 479-500; MR 16, 990]. Let A, B be binary operations in a set M ; A is left-distributive (symbolically, $A dB$) with respect to B provided

$$A(x, B(y, z)) = B(A(x, y), A(x, z))$$

for every $x, y, z \in M$. Let Π be a system of binary operations in M ; $(M; \Pi)$ is a semi-distributive system (SDS) provided for every $A \in \Pi$ there are $X, Y \in \Pi$ satisfying $X d A, A d Y$; e.g., if G is a commutative group and

$$\Pi = \{ax^i y^j | a \in G; i, j \in D; D = \{0, \pm 1, \pm 2, \dots\}\},$$

then (G, Π) is an SDS, because $ax^i y^j d ex^i y^{j-1} \dots x^i y^0$ (e denoting the unit of G). Theorem 2.1: If $\Pi = \{A, B, C\}$, $A dB, B d C$; if f, e are neutral elements with respect to B and C , respectively; if B is cancellable on the right side by every element $\neq e$; if C is cancellable on the left side by every element of M ; then for a given $a \in M$ one has $A(a, e) = e$ or $A(a, x) = f$ for every $x \in M$. Theorem 2.3: If $(M; A)$ is a group and $A dB, B d A$, then $B(x, y) = y$ for every $x \in M$. Various examples are given.

Đ. Kurepa (Zagreb)

5742:

Neumer, Walter. Kritische Zahlen und bestimmt divergente transfinite Funktionen. *Math. Z.* 70 (1958), 190-192.

A theorem of Fodor [Acta Sci. Math. Szeged 17 (1956), 139-142; MR 18, 551] in an equivalent form is proved.

F. Bagemihl (Notre Dame, Ind.)

COMBINATORIAL ANALYSIS

5743:

Mouette, Léon. Recherches sur la théorie des triades. *Mathesis* 66 (1957), 283-287.

Steiner triple systems are constructed, with emphasis

on $N=15$. In this case the author can give constructions for over 40 systems. [For $N=15$ an exhaustive listing includes 80 systems. See Hall and Swift, *Math. Tables Aids Comput.* 9 (1955), 146-152; MR 18, 192.]

H. J. Ryser (Columbus, Ohio)

5744:

Levine, Jack. A binomial identity related to rhyming sequences. *Math. Mag.* 32 (1958), 71-74.

A rhyming sequence (=rhyme scheme) is obtained by marking all rhyming syllables with the same letter or number; thus for n verses and r different rhymes indicated by numbers $1, 2, \dots, r$ with rhyme i occurring n_i times, $n = n_1 + n_2 + \dots + n_r$, the sequence is an arrangement of these numbers on a line such that the first appearances of the numbers are in the natural order. The number of such arrangements has been given by W. D. Smith [solution to problem 3731, *Amer. Math. Monthly* 56 (1949), 40] as

$$A(n_1, n_2, \dots, n_r) = \prod_{i=1}^r \binom{n_i + n_{i+1} + \dots + n_r - 1}{n_i - 1}.$$

The binomial identity in question is obtained by classifying these arrangements according to the number in the last position, which gives

$$A(n_1, n_2, \dots, n_r) = A(n_1 - 1, n_2, \dots, n_r)$$

$$+ A(n_1, n_2 - 1, \dots, n_r) + \dots + A(n_1, n_2, \dots, n_r - 1).$$

It is also shown that the sum of $A(n_1, n_2, \dots, n_r)$ over all r -part compositions of n is the Stirling number of the second kind, $S(n, r)$, as is required.

J. Riordan (New York, N.Y.)

ORDER, LATTICES

5745:

Benado, Mihail. Sur la théorie générale des ensembles partiellement ordonnés. *C. R. Acad. Sci. Paris* 247 (1958), 2265-2268.

First comes a generalization of the introduction of lattice operations into a partially ordered set L (essentially along the lines of earlier work of the author [Czechoslovak Math. J. 5(80) (1955), 308-344; MR 17, 937], except that all majorants, instead of only minimal ones, are considered, and dually). For a set $X = \{x_i\}$ of elements, $VY = V\{x_i\}$ is defined as the set of all majorants of X ; for a set $\{X_i\}$ of sets, VX_i is defined as the set union of all VY_j , where Y_j contains one element from each X_i . Simple properties are stated. A binary relation γ between elements and subsets of L is called a division relation in L if $d\gamma A$ implies $d \in VA$; dually (retaining set union) for multiplication relations. Several examples, sub-categories, and elementary properties are given. L is said to have an analytic division structure if there exist division and multiplication relations γ and σ such that: (1) if $a \geq b$ in L then there exist $d, m \in L$ with $d\gamma\{a, b\}$ and $m\sigma\{a, b\}$; (2i) if also $x \in d/m$ then there exists $a_1 \in L$ such that $d \geq a_1\gamma\{x, a\}$; and (2') dually. L has a synthetic division structure if (1) holds and if $a_1 \in d/a, b_1 \in d/b$ imply the existence of $m_1 \in L$ with $m \leq m_1\sigma\{a_1, b_1\}$ and dually. Any partially ordered set has at least one division structure which is both analytic and synthetic, namely $d\gamma A$ if and only if $d \in VA$ and dually.

P. M. Whitman (Baltimore, Md.)

5746:

Bosák, J. Generalization of the method of complete induction. *Acta Fac. Nat. Univ. Comenian. Math.* 2 (1958), 255-256. (Czech)

The author gives five statements in connexion with the principles of induction in sets of real numbers and, more generally, in linearly ordered sets. Only O. Perron [*Jber. Deutsch. Math. Verein* 35 (1926), 194-203] is cited. All the results are contained in the reviewer's thesis: *Ensembles ordonnés et ramifiés*, Paris 1935; in particular, pp. 21-25 [also published in *Publ. Math. Univ. Belgrad* 4 (1935), 1-138]. Cf. the reviewer's results with respect to partially ordered sets [*Acad. Serbe Sci. Publ. Inst. Math.* 8 (1955), 1-12; *C. R. Acad. Sci. Paris* 242 (1956), 2202-2203; *MR* 17, 1065]. *Đ. Kurepa* (Zagreb)

5747:

Kwan, Chao-chih. Quelques remarques sur les topologies définies sur un treillis. *Advancement in Math.* 3 (1957), 662-669. (Chinese. French summary)

The author proves in this paper a few results concerning two questions posed by G. Birkhoff [*Lattice theory*, Amer. Math. Soc. Colloq. Publ., New York, 1948; *MR* 10, 673]. The first proposition states that every closed ideal of a lattice L is the intersection of principal ideals if and only if L has a largest element I . This answers problem 24 of G. Birkhoff, and, at the same time, shows that this problem is trivial. The second proposition states that an ideal of a complete lattice L is closed in the topology (O) or in the interval topology if and only if it is a closed ideal of the lattice L . The third proposition states that every ideal of a lattice L is a principal ideal if and only if the lattice L is complete and every element of L is inferiorly compact. Here, an element $a \in L$ is said to be inferiorly compact if, for every relation $a = \bigvee_{i \in J} x_i$, where J is any set of indices, there exists a finite subset F of J such that $a = \bigvee_{i \in F} x_i$. The fourth proposition states that an element a in a lattice L is an isolated point in the topology (O) if and only if this element a is inferiorly and superiorly compact. This answers problem 21(a) of G. Birkhoff. *Sze-tsen Hu* (Detroit, Mich.)

5748:

Vaida, Dragoș. Détermination des structures modulaires par des systèmes à axiomes indépendants. *Acad. R. P. Romîne. Stud. Cerc. Mat.* 8 (1957), 457-466. (Romanian. Russian and French summaries)

Consider for a set S binary operations which satisfy: (I_1) $x[(x+y)+z]=x$; (I_2) If $x=z$ or if, for some w , we have one of the relations $x=wz$, $x=zw$, or $z=w+x$, then $z(y+x)=x+yz$.

The author as one of his two main theorems (the other concerns operations on partly ordered sets) shows S is a modular lattice if and only if (I_1) and (I_2) hold. This characterization compares, in compactness, with that of Kolibiar [*Czechoslovak Math. J.* 6 (81) (1956), 381-386; *MR* 21 #1278]. *M. Sholander* (Pittsburgh, Pa.)

5749:

Dwinger, Ph. Some theorems on universal algebras. III. *Nederl. Akad. Wetensch. Proc. Ser. A* 61=Indag. Math. 20 (1958), 70-76.

[For parts I, II see same Proc. 60 (1957), 182-195; *MR* 19, 240.] The author presents an obvious gener-

alization to modular lattices of a theorem of Zassenhaus on characteristically simple groups.

R. P. Dilworth (Pasadena, Calif.)

GENERAL ALGEBRAIC SYSTEMS

See also 5749, 5817.

5750a:

Fujiwara, Tsuyoshi. Supplementary note on free algebraic systems. *Proc. Japan Acad.* 33 (1957), 633-635.

5750b:

Fujiwara, Tsuyoshi. Note on free products. *Proc. Japan Acad.* 33 (1957), 636-638.

The first note is a supplement to a previous paper [same Proc. 32 (1956), 662-664; *MR* 18, 636] on general algebraic systems in the sense of Shoda. The author determines certain conditions for the existence of free systems with respect to a given class of relations.

The second note deals with the definition of a free product and the condition that a system be contained in such a product. *O. Ore* (New Haven, Conn.)

5751:

Sade, A. Groupoides orthogonaux. *Publ. Math. Debrecen* 5 (1958), 229-240.

The author calls a system with two operations (\cdot) and (\times) orthogonal when

$$x \cdot y = z \cdot t, \quad x \times y = z \times t$$

implies $x=z$, $y=t$. An often used example is

$$x \cdot y = ax + by, \quad x \times y = cx + dy$$

which is orthogonal when $ad-bc \neq 0$. The author points out how new orthogonal systems may be derived from given ones by forming conjoinings, composition of operations and various transforms. A condition is given for a groupoid to permit an orthogonal operation.

O. Ore (New Haven, Conn.)

THEORY OF NUMBERS

See also 5840, 5895, 5994, 6340.

5752:

Moessner, Alfred. Ein Diophantisches Problem. *Euclides, Madrid*, 17 (1957), 115-120.

The author gives some solutions of the simultaneous system $\sum_{i=1}^n x_i^n = \sum_{i=1}^n y_i^n$ for $n=1, 2, 3$ and for $n=1, 2, 3, 4$. *W. Ljunggren* (Blindern)

5753:

Jacobsthal, Ernst. Über einige Sätze der elementaren Zahlentheorie. *Norske Vid. Selsk. Forh., Trondheim*, 31 (1958), no. 15, 8 pp.

Some elementary theorems on the nature of the distinct residue classes (mod n) represented by the elements of an arithmetic progression. *L. Moser* (Edmonton, Alta.)

5754:

Comment, P. On the identity $f(g \wedge h) = (fg) \wedge (fh)$. *Riveon Lematematika* 11 (1957), 39-40. (Hebrew. French summary)

The author assumes as known the papers by D. Yarden

and Th. Motzkin and by N. Kabaker [Yarden and Motzkin, same Riveon 1 (1946), 1-7; MR 8, 316; Kabaker, *ibid.* 1 (1946), 29; MR 8, 316] and uses their notations. He proves that if f is a multiplicative function with the property $f(p^2) \neq f^2(p)$ for every prime p , then $g=\delta$ (Dirichlet's unit) is the only strongly multiplicative function which satisfies the identity $f(g^{\wedge}h) = (fg)^{\wedge}(fh)$ for every natural n and every multiplicative h , without f being strongly multiplicative. B. A. Amirà (Jerusalem)

5755:

McCarthy, P. J. A congruence property of Ramanujan's function. *Quart. J. Math. Oxford Ser. (2)* 8 (1957), 141-142.

Ramanujan's function $\tau(n)$ is defined by

$$\sum_1^n \tau(n)x^n = x \prod_1^\infty (1-x^n)^{24} \quad (|x| < 1).$$

Using results of Ramanujan [Collected Papers, University Press, Cambridge, 1927] the author proves

$$\tau(n) \equiv 26\sigma_3(n) + 3\sigma_7(n) + 21\sigma_{11}(n) + 34U_3,7(n) \pmod{49},$$

where $U_{r,s}(n) = \sum_{k=1}^{n-1} \sigma_r(k)\sigma_s(n-k)$ ($n > 1$), $U_{r,s}(1) = 0$, and $\sigma_a(n) = \sum_{d|n} d^a$. This is simpler than a result of Bambah and Chowla [Math. Student 14 (1946), 24-26; MR 9, 411]. When $n \equiv 3, 5, 6 \pmod{7}$, the residue of $\tau(n)$ (mod 49) is very simply expressed by $\sigma_a(n)$, a result of D. H. Lehmer. S. Chowla (Boulder, Colo.)

5756:

Cohen, Eckford. Representations of even functions (mod r). I. Arithmetical identities. *Duke Math. J.* 25 (1958), 401-421.

In an earlier paper [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 939-944; MR 17, 713] the author introduced even functions and gave representations of such functions with the help of Ramanujan sums. Now he gives these expansions in another form suitable for applications. He gets a new proof of a theorem of Anderson and Apostol concerning even functions satisfying a Hölder type relation and also gets an inversion formula for even functions. Further, he introduces special classes of even functions and characterizes them trigonometrically.

H. Bergström (Göteborg)

5757:

Cohen, Eckford. Congruence representations in algebraic number fields. II. Simultaneous linear and quadratic congruences. *Canad. J. Math.* 10 (1958), 561-571.

[For part I, see Trans. Amer. Math. Soc. 75 (1953), 444-470; MR 15, 508]. Let F be a finite algebraic extension of the rational number field and P an ideal in F of norm p^l where p is odd. The author considers the pair of congruences

$$m \equiv \alpha_1 x_1^2 + \cdots + \alpha_s x_s^2, \quad n \equiv \beta_1 x_1 + \cdots + \beta_s x_s \pmod{P^l},$$

where m, n, α_i and β_i are integers of F and α_i and β_i prime to P . By the help of simple functions of the constants m, n, α_i and β_i he gives explicitly the number $N_s(m, n)$ of simultaneous solutions of these congruences. The cases when $N_s(m, n) = 0$ are separately described. For the computation of $N_s(m, n)$ generalized Cauchy-Gauss sums are used.

H. Bergström (Göteborg)

5758:

Maier, Wilhelm. Über einige Lambertsche Reihen. *Arch. Math.* 9 (1958), 186-190.

For $\text{Im}(\omega) > 0$, $\log i = \pi i/2$, put

$$H(\omega) = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{\eta^{s-1}}{e^{-2\pi i n \omega} - 1},$$

an analytic function of ω and s . Then

$$H(\omega) = \sum_{h=-\infty}^{\infty} \sum_{k=1}^{\infty} (h+k\omega)^{-s},$$

and for $\text{Re}(s) > 2$, $|\text{arc}(\omega/i)| < \pi/2$,

$$H(\omega) = - \frac{(-2\pi i)^s}{\Gamma(s)} \int_{(s+\pi)} \frac{\Gamma(t)\zeta(t)\zeta(t+1-s)}{(-2\pi i \omega)^s} dt.$$

It is shown that for $\omega = p/q + iy$, where $(p, q) = 1$,

$$(*) \quad H(\omega) = \frac{\zeta(s)}{(q\omega - p)^s} + \frac{\zeta(2-s)}{q\omega - p} \frac{(-2\pi i q)^{s-1}}{2\Gamma(s)} - \frac{\zeta(1-s)(-2\pi i)^s}{2\Gamma(s)} \sum_{r=1}^{q-1} \zeta\left(1-s, \frac{r}{q}\right) \cot \frac{r\pi p}{q} + O(q\omega - p),$$

where $\zeta(s, a)$ is the Hurwitz zeta-function. From (*) the author derives

$$\lim_{\delta \rightarrow 0} \frac{-\delta}{\log \delta} \int_0^{\infty} e^{-\delta t} |\zeta(\tfrac{1}{2} + it)|^2 \cos\left(t \log \frac{p}{q}\right) dt = (pq)^{-1}$$

for $(p, q) = 1$, which reduces to a formula of Hardy for $p = 1$. Making use of the representation

$$(-2\pi i)^{-s} \Gamma(s) H(\omega, s) = \sum_{k=1}^{\infty} e^{2\pi i k \omega} k^{s-1} \sigma_{1-s}(k), \quad \sigma_{1-s}(k) = \sum_{d|k} d^{1-s},$$

the author obtains the integral equation

$$\left(\frac{a+b}{a}\right) \sum_{n=1}^{\infty} q_{a,b}(n) H(n\omega, a+b-1) = \frac{1}{2\pi i} \int_{\delta}^{\delta+1} H(t, a+1) H(\omega-t, b+1) dt,$$

where $0 < \text{Im}(\delta) < \text{Im}(\omega)$ and

$$\frac{\zeta(s-a)\zeta(s-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} n^{-s} q_{a,b}(n).$$

L. Carlitz (Durham, N.C.)

5759:

Wiener, Norbert; and Wintner, Aurel. Notes on Pólya's and Turán's hypotheses concerning Liouville's factor. *Rend. Circ. Mat. Palermo (2)* 6 (1957), 240-248.

Pólya pointed out that if

$$(P) \quad \sum_1^x \lambda(n) \leq 0 \quad (x > 1),$$

where $\lambda(n)$ is Liouville's function, then the Riemann hypothesis (R.H.) is true (however, Haselgrove has informed the reviewer that he has disproved Hypothesis (P)). Turán has observed that if

$$(T_1) \quad \sum_1^x \frac{\lambda(n)}{n} \geq 0 \quad (x > x_0),$$

then the R. H. is true, and also that the R. H. is true if

$$(T_2) \quad \sum_1^x n^{-s} \neq 0 \quad (\Re(s) > 1) \text{ for } x > x_0.$$

The authors prove the following result related to (T_2) : If there exists a positive $\varepsilon < 1$ for which

$$(W) \quad \sum_1^{\infty} \frac{r^n}{n^s} \neq 0 \quad (\Re(s) > 1, 1-\varepsilon < r < 1),$$

then the R. H. is true. The authors also observe that in view of a result of Showalter [in Bateman and Chowla, *J. Indian Math. Soc. (N.S.)* 17 (1953), 177-181; MR 15, 939] the hypothesis:

$$(H) \quad \text{For all } x, \sum_1^x \frac{\chi(n)}{n} \geq 0 \quad (x \geq 1)$$

for all real characters χ , would also imply the R. H.
S. Chowla (Boulder, Colo.)

5760:

Cudakov, N. G.; and Bredihin, B. M. Application of Parseval's equality for the estimation of sum functions of characters of numerical semigroups. *Ukrain. Mat. Ž.* 8 (1956), 347-360. (Russian)

The article deals with generalized characters, i.e., multiplicative functions defined on a multiplicative semigroup of real numbers. In connection with each character χ there is considered a summation function of the character

$$H(x) = \sum_{\alpha \leq x} \chi(\alpha) \text{ and } \sum \chi(\alpha) \alpha^{-s},$$

namely, the L -function of the character.

The author obtains various estimates from below (Ω -estimates) for summation functions of characters on the basis of the Parseval equality:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{L(\sigma + it)}{\sigma + it} \right|^2 dt = \int_0^{\infty} |H(x)|^2 e^{-2\sigma x} dx.$$

N. I. Romanov (RŽMat 1958 #1788)

5761:

Hawkins, David. The random sieve. *Math. Mag.* 31 (1957/58), 1-3.

The author investigates the following sieve: Check the number 2, then with probability $\frac{1}{2}$ strike out each subsequent number. If P_2 is the first number not stricken out, check it and strike out each number thereafter with probability $1/P_2$. P_3 is the next number not stricken out, we check it and strike out each subsequent number with probability $1/P_3$, etc. Thus we obtain a random sequence $P_1 = 2 < P_2 < P_3 < \dots$. The author outlines a proof that with probability 1, $P_n/n \log n \rightarrow 1$ [further literature: Cramer, *Acta Arith.* 2 (1936), 23-46].

P. Erdős (Birmingham)

5762:

Hugot, Marthe; et Pisot, Charles. Sur certains entiers algébriques. *C. R. Acad. Sci. Paris* 246 (1958), 2831-2833.

"Un entier algébrique α dont tous les conjugués autres que lui-même sont en module ≤ 1 est zéro d'un polynôme dont les coefficients sont des entiers rationnels de valeur absolue inférieure à α ."

Authors' summary

5763:

Tatuzawa, Tikao. On the Waring problem in an algebraic number field. *J. Math. Soc. Japan* 10 (1958), 322-341.

Let F be an algebraic extension of the rationals of degree n with r_1 and $2r_2$ complex conjugate fields. Let J_k denote the ring generated by the k th powers of integers in F . Using his extension of the Hardy-Littlewood circle method as modified by Vinogradoff, C. L. Siegel [*Amer. J. Math.* 66 (1944), 122-136; *Ann. of Math.* (2) 46 (1945), 313-339; *MR* 5, 200; 7, 49] proved the following theorem. If r is a totally positive integer in J_k and $s > nk(2^{k-1} + n)$, then the equation

$$v = \xi_1^k + \xi_2^k + \dots + \xi_s^k$$

is solvable for $N(v)$ sufficiently large and subject to the conditions $|\xi_0^{(i)}|^k < |v^{(i)}|$ for $i = r_1 + 1, \dots, r_1 + r_2$ and ξ_i totally non-negative. In addition, he gave an asymptotic formula for the number of representations. The bound on s is the analogue of the bound obtained by Hardy and Littlewood for the rational case and involves an exponential factor. This left open the possibility of im-

proving the lower bound on s in accordance with the methods of Vinogradoff. In this paper, the author, using these methods, replaces the lower bound by $8nk(k+n)$.

The general method is substantially that developed by Siegel for the Waring problem over algebraic number fields, but more careful estimates are required throughout and the devices of Vinogradoff are employed. In addition, an account of the singular series is given with the help of valuation theory. R. Ayoub (University Park, Pa.)

5764:

Urazbaev, B. M. On the least discriminant of an Abelian field. *Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh.* no. 6(10) (1957), 108-111. (Russian. Kazah summary)

The author first gives the general expression for the discriminant of an abelian field over the rationals with prime-power galois group of given type, which is furnished by the Führerdiskriminantenformel [e.g., H. Hasse, *J. Fac. Sci. Univ. Tokyo* 2 (1934), 477-498]. The remark that a prime congruent to 1 modulo another prime is greater than it then furnishes an estimate below for the discriminants of the fields under discussion. The author appears to use the symbol O in estimates where Ω is customary. J. W. S. Cassels (Cambridge, England)

5765:

Carlitz, L. A note on the irrational modular equation of order seven. *Nieuw Arch. Wisk.* (3) 5 (1957), 143-145.

The author deduces by elementary means from the irrational modular equation of order 7 that

$$2 \frac{\varphi_r(q)}{\psi_r(q)} = - \frac{\psi(q)}{\varphi(q)} \quad (1 \leq r \leq 6),$$

where

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{1+q^{2n}}{1+q^{2n-1}} = \sum_{n=0}^{\infty} a(n)q^n,$$

$$\psi(q) = \prod_{n=1}^{\infty} \frac{1-q^{2n-1}}{1+q^{2n-1}} = \sum_{n=0}^{\infty} b(n)q^n,$$

$$\varphi_r(q) = \sum_{n=0}^{\infty} a(7n+r-1)q^n,$$

$$\psi_r(q) = \sum_{n=0}^{\infty} b(7n+r)q^n.$$

The modular equation of order 23 is also considered and yields a more complicated result.

R. A. Rankin (Glasgow)

5766:

Foster, D. M. E. On a class of quadratic polynomials in n variables. *Quart. J. Math. Oxford Ser.* (2) 9 (1958), 241-256.

Let L_1, \dots, L_n denote n homogeneous linear forms in n variables u_1, \dots, u_n with real coefficients and determinant $\Delta \neq 0$, and let c be any real number. Let $q = \pm L_1^2 \pm \dots \pm L_{n-1}^2$ and $p = q + L_n + c$; let q be of rank $n-1$ and signature $s = 2r+1-n$, where r is the number of positive signs in the expression above for q . The author proves the following two theorems. Theorem 1: If the coefficients of $p-c$ are not all in a rational ratio, p assumes arbitrarily small values for integers u_1, \dots, u_n . Theorem 2: If the coefficients of $p-c$ are in a rational ratio, then there are integers u_1, \dots, u_n satisfying $|p| \leq (\frac{1}{2}|\Delta|)^{2/(n+1)}$ except possibly when $|s| = n-1$ and $n \geq 10$. The second theorem is an improvement of a result of Macbeath.

B. W. Jones (Boulder, Colo.)

5767:

van der Waerden, B. L. Ein diophantisches Problem von O. Perron. Arch. Math. 9 (1958), 54-58.

Let $f(x) = |a-x||b-x|$, where a, b are real numbers and x is a rational integral variable. Put $a = d - \frac{1}{2}c$, $b = d + \frac{1}{2}c$, and let $M(c, d) = \min f(x)$ over all x and let $G(c) = \max M(c, d)$ over all real d . Perron [Math. Z. 67 (1957), 176-180; MR 19, 124] has shown that $G(c) \leq 1$ for $c^2 \leq 8$, equality holding when $c^2 = 5, 8$. The problem of determining $G(c)$ is now solved; the graph of $G(c)$ as a function of c^2 being piecewise linear. In particular, $G(c) > 1$, except when $c^2 \leq 8$ or $12 \leq c^2 \leq 13$.

J. H. H. Chalk (Hamilton, Ont.)

5768:

Woods, A. C. On two-dimensional convex bodies. Pacific J. Math. 8 (1958), 635-640.

Consider a lattice L of points in E^2 with coordinates x^1, x^2 . For a given closed convex set K with interior points let $\mu_i(K)$ ($i=1, 2$) be the supremum of the numbers c_i such that $c_i K$ contains at most $i-1$ independent points of L . Put $\Delta(K) = \inf d(L')$, where $d(L')$ is the determinant of L' , and L' traverses all lattices such that K contains at most one point of L' . It is proved here that $\mu_1(L)\mu_2(L) \times \Delta(K) \leq d(L)$, for any K and L . If $K(t)$ denotes the subset $|x^2| \leq t$ of K then $\Delta(K(t))/t$ decreases for $t > 0$. These results were proved previously for K with center, respectively, by C. Chabauty [Ann. Sci. Ecole Norm. Sup. (3) 66 (1949), 367-394; MR 11, 418] and by Mahler [Rend. Mat. e Appl. (5) 13 (1954), 38-41].

H. Busemann (Cambridge, Mass.)

5769:

Schmidt, Wolfgang. Mittelwerte über Gitter. II. Monatsh. Math. 62 (1958), 250-258.

In the first section the author corrects the proof of theorem 2 of part I [Monatsh. Math. 61 (1957, 269-276; MR 20#1672]. In the second part he evaluates the integral

$$\int \Sigma f(Ag_1, \dots, Ag_n) d\mu(A),$$

where μ is Siegel's measure in the group L of all linear transformations of determinant 1, where the sum is taken over all sets g_1, \dots, g_n of vectors which form a basis for the lattice of points with integral coordinates, and where the integral is taken over a fundamental region in the group L relative to the sub-group of all integral unimodular matrices.

C. A. Rogers (Birmingham)

COMMUTATIVE RINGS AND ALGEBRAS

See also 5793, 6045.

5770:

*Lugowski, Herbert; und Weinert, Hanns Joachim. Grundzüge der Algebra, Teil II. Allgemeine Ring- und Körpertheorie. Mathematisch-Naturwissenschaftliche Bibliothek, Bd. 10. B. G. Teubner Verlagsgesellschaft, Leipzig, 1958. 250 pp. (1 insert) DM 11.00.

For Teil I (1957) see MR 19, 728. The present Teil II contains chapters V through VIII, with the headings: Grundbegriffe der Ring- und Körpertheorie, Ringkonstruktionen, homomorphe Abbildungen, Algebraische Strukturen mit Operatorbereichen, Teilbarkeitslehre, ZPE-Ringe. In this volume there are also numerous exercises with complete solutions.

5771:

Szele, Tibor. An elementary proof of the fundamental theorem for finite fields. Mat. Lapok 7 (1956), 249-254. (Hungarian. Russian and English summaries)

The author gives a very simple proof of the following classical theorem: The number of elements in a finite field is a power of a prime. For any prime p and positive integer n there exists a finite field with p^n elements. Any two finite fields with the same number of elements are isomorphic.

A. Kertész (Debrecen)

5772:

Fielder, Daniel C. A note on summation formulas of powers of roots. Math. Tables Aids Comput. 12 (1958), 194-198.

If S_b is the sum of the b th powers of all of the roots of the polynomial equation $f(x)=0$, where $f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n$, then a theorem of Newton gives the recursion formula

$$ka_k + \sum_{i=1}^k S_i a_{k-i} = 0, \quad k=1, 2, \dots,$$

which may be used for the iterative determination of S_b . This note presents an explicit summation formula for S_b , involving many repetitive combinations of the a 's. Combinatorial rules are stated to determine the summation formula for S_b involving the minimal number of summands and the results are tabulated for $b=1, 2, \dots, 11$.

G. L. Walker (Southbridge, Mass.)

ALGEBRAIC GEOMETRY

See also 6114.

5773:

*Hauser, Wilhelm; und Burau, Werner. Integrale algebraischer Funktionen und ebene algebraische Kurven. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. 103 pp. (12 inserts) DM 9.20.

This book is an introduction to the geometry of plane algebraic curves; the methods used are those of elementary algebra. General results, such as Bézout's theorem and Plücker's equations, are included and illustrated by means of aptly chosen examples. The genus of a curve is mentioned, but the proof of its birational invariance is outside the scope of the book. The process of representing a rational curve parametrically is explained; and the usefulness of such a representation as a means of evaluating Abelian integrals on a rational curve is stressed.

D. Kirby (Leeds)

5774:

Morelock, J. C.; and Perry, N. C. A note concerning homogeneous polynomials. Math. Mag. 31 (1957/58), 75-79.

Il s'agit de trouver toutes les équations homogènes

$$F(x_1, x_2, x_3, x_4) = 0,$$

de degré p , qui restent invariants dans la transformation

$$T: (x_1', x_2', x_3', x_4') = (x_1, Ex_2, E^2x_3, E^3x_4)$$

où p est un nombre premier et $E^p = 1$. Il faudra que les termes $x_1^a x_2^b x_3^c x_4^d$ du polynôme F se transforment en acquérant un facteur E^s dont l'exposant s est le même

pour tous les termes. On écrit et on résout très facilement en nombres entiers les équations exprimant cette propriété en supposant connus, par exemple, les valeurs de a et de d . Les auteurs donnent les formules résolutives générales et ils les appliquent au cas $p=5$ (on a dans ce cas 5 solutions).
E. G. Togliatti (Genoa)

5775:

Bilo, Julien. Sur une transformation quadratique. *Mathesis* 66 (1957), 176-182.

The author uses a special type of plane quadratic transformation in which the two images point are collinear with a center S , and the locus of each of a pair of image points is a cubic of a net. Having defined the Gergonne point Γ of a conic γ inscribed in a triangle ABC as the reciprocal of the anticomplement of the center I of γ , he shows that if I describes a cubic Q through the center of gravity G of the triangle ABC and through the vertices of the complementary triangle $A'B'C'$ (the tangents at the vertices being the corresponding medians $A'A, B'B, C'C$), the Gergonne point Γ describes a cubic Q_1 through G and the vertices of both ABC and $A'B'C'$. Since the image points I, Γ are always collinear with a fixed point S , this is a quadratic transformation of the type described. S lies on both Q and Q_1 . Also, the line SG is a common tangent to Q and Q_1 at G . Other properties are obtained and special cases discussed.

T. R. Holcroft (Aurora, N.Y.)

5776:

Godeaux, Lucien. Sur le lieu des droites des surfaces cubiques d'un faisceau. *Mathesis* 66 (1957), 249-252.

Let F be a pencil of cubic surfaces without multiple points for which the basis curve C of order 9 is non-singular and irreducible. Assume further that the tangent planes of the surfaces of F are distinct at each point of C . A surface of F contains 27 lines each of which is a trisecant of C . It is proved that the locus of the lines of the cubic surfaces of F is a ruled surface R of order 42 and genus 70, containing C as an 11-fold multiple curve and having a double curve Γ of order 255 which intersects C in 630 points. F contains 32 surfaces with a double point. Each of these double points is a multiple point of order 6 on R and of order 15 on Γ .

T. R. Holcroft (Aurora, N.Y.)

5777:

Dominik, Palman. Über eine Flächenart 3. Ordnung mit vier Doppelpunkten sowie über zirkuläre Kurven 3. Ordnung. *Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke* 6 (302) (1957), 145-170. (Serbo-Croatian. German summary)

Sind die zugeordneten Punkte einer räumlichen Transformation auf den Strahlen einer linearen Kongruenz H konjugiert bezüglich einer Fläche Ψ zweiter Ordnung gelegen, so handelt es sich um eine kubische Inversion, in welcher einer Ebene Φ eine allgemeine Fläche Φ_{Ψ^3} dritter Ordnung entspricht. Werden die Leitgeraden p_1 und p_2 der (linearen hyperbolischen) Kongruenz H so gewählt, daß sie konjugierte Polaren der Fläche Ψ sind, so entspricht in der zugehörigen kubischen Inversion einer Ebene Ψ eine allgemeine Fläche Φ_{Ψ^3} dritter Ordnung mit vier Doppelpunkten. Je nach Wahl der Fläche Φ erzielt Verfasser durch derartige kubische Inversionen verschiedene interessante Flächen Φ_{Ψ^3} . Zuerst wird für Ψ ein Rotationsparaboloid gewählt und dann ein Rotationszylinder. Für die Fläche $\Phi_{\Psi^3}=\Omega$ ergeben sich im ersten Falle zwei reelle und zwei konjugiert-imaginäre Doppelpunkte. Die Leitgeraden p_1 und p_2 sind dann

reelle vierdeutige Geraden der Fläche. Die Geraden, welche die reellen Doppelpunkte K und L_{∞} (in den Scheiteln des Paraboloids) mit den Doppelpunkten auf der unendlich fernen Geraden p_2 verbinden, sind zwei Paare isotroper vierdeutiger Geraden der Fläche. Außer diesen Geraden gibt es auf solchen Flächen eine reelle eindeutige Gerade h und ein Paar eindeutiger konjugiert-imaginärer Geraden. h und p_2 bestimmen eine Ebene Π . Verfasser betrachtet dann die Schnittkurve (ρ, Ω) einer beliebigen Ebene ρ des Raumes mit Ω und projiziert diese aus L_{∞} auf Π . Dabei entsteht eine zirkuläre Kurve dritter Ordnung. Im Falle eines Rotationszylinders Ψ wird p_1 als Zylinderachse gewählt, p_2 liegt dann in der uneigentlichen Ebene und ist senkrecht zu p_1 . Φ entspricht jetzt durch kubische Inversion eine Fläche dritter Ordnung Ω , mit zwei Doppelpunkten auf p_2 und einem zweideutigen Doppelpunkt L_{∞} (uneigentliche Spitze des Zylinders), Ω ist daher ein Spezialfall von Ω . Als Projektion auf die Ebene Π ergeben sich hier durch analoge Konstruktionen ein System von Strophoidalen dritter Ordnung. Weiterhin gewinnt Verfasser auf diesem Wege auch einige Sätze aus der Theorie der zirkulären Kurven dritter Ordnung, insbesondere auch Schmiegekreiskonstruktionen in absoluten Punkten einer zirkulären Kurve.

M. Pinl (Köln)

5778:

Pompili, Giuseppe. Sul genere geometrico delle superficie algebriche irregolari. *Rend. Mat. e Appl.* (5) 17 (1958), 210-230.

Sia F una superficie algebrica di generi p_g, p_a , priva di fasci irrazionali e d'irregolarità $q \neq 2$. Essa è razionalmente equivalente ad una superficie F^* della sua varietà di Picard, appartenente ad un sistema continuo ∞^d con $d \geq 3q-2$ di superficie tra loro birazionalmente equivalenti. Poichè se una superficie appartiene, sopra una varietà di dimensione p e di genere geometrico positivo, ad un sistema continuo ∞^d risulta: $p_g \geq d-p+3$, e d'altra parte il genere geometrico di F^* non supera quello di F , ne segue: $p_g \geq 2q+1$, ossia: $p_g \leq 2p_a-1$. Questo risultato ed un teorema di G. Dantoni forniscono in ogni caso un confine inferiore per il genere geometrico di una superficie di data irregolarità. Tra le più significative conseguenze della disuguaglianza $p_g \leq 2p_a-1$ si ha che le superficie con $p_g \geq 2p_a$ hanno $p_g = p_a = 0$, oppure hanno irregolarità $q=2$, oppure son dotate di fascio irrazionale. Questo risultato, insieme alla formula di Picard-Alexander ed alla disuguaglianza di Severi: $\rho_0 \geq 2p_g$, fornisce alcune indicazioni sulle superficie con serie di Severi d'ordine basso.

D. Gallarati (Genoa)

5779:

d'Orgeval, B. Construction de surfaces irrégulières. *Publ. Sci. Univ. Alger. Sér. A* 3 (1956), 125-142.

Ce mémoire se rattache à des études récentes de P. Burniat ayant le but de construire des surfaces algébriques irrégulières sous la forme de plans quadruples et octuples abéliens [voir P. Burniat, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* (8) 16 (1954), 326-331, 459-463; *MR* 16, 615]. Après avoir rappelé rapidement le procédé de P. Burniat, pour la construction d'un plan multiple abélien d'ordre 2^n en partant de n radicaux donnés $\sqrt[n]{f_i(x)}$ qui permettent de définir les courbes de diramation E_1, E_2, \dots, E_n , l'A. répète le cas $n=3$ de P. Burniat, puis il développe complètement les cas $n=4, 5, 6, 7$, en déterminant pour chaque cas les valeurs des genres géométrique, arithmétique, linéaire, etc. En général, on trouve que pour chaque valeur de n il existe $n-1$

familles de surfaces dépendant d'un paramètre r , qui se peuvent représenter sur un plan multiple abélien d'ordre 2^n ; les surfaces d'une même famille ont toutes l'irrégularité $1 + (i-2)2^{i-2}$ ($2 \leq i \leq n$); sur chacune de ces surfaces il existe un faisceau irrationnel de courbes dont le genre est égal à l'irrégularité de la surface, tandis que le genre des courbes du faisceau a la valeur $1 + 2^{n-1}(r-2)$.

E. G. Togliatti (Genoa)

5780:

Rosenlicht, Maxwell. Extensions of vector groups by abelian varieties. Amer. J. Math. 80 (1958), 685-714.

The author develops the theory of simple differentials of the second kind on an algebraic variety by studying the set of all abelian group extensions of direct products of additive groups over a given abelian variety. The set of equivalence classes of extensions of the additive group of one dimension by a given abelian variety A has the structure of a vector space. The chief results of the paper, established in theorems 1 and 3, are that this vector space has the same dimension as A and is isomorphic to the cohomology space $H^1(A, O_A)$, in which O_A is the sheaf of local rings on A . All of the methods used are purely algebraic, except in the part of proposition 11 dealing with the case of characteristic zero.

T. R. Hollcroft (Aurora, N.Y.)

LINEAR ALGEBRA

See also 6068.

5781:

Molinaro, I. Sur les endomorphismes de Reynolds de fonctions définies sur un ensemble fini. Publ. Sci. Univ. Alger. Sér. A 4 (1957), 87-101.

Let E be a given set, and let \mathcal{A} be a ring of functions defined over E and valued in a totally ordered field \mathcal{K} . According to M. L. Dubreil-Jacotin [Colloque d'algèbre supérieure, Bruxelles, 1956, pp. 9-27, Centre Belge de Recherches Math., Etablissement Centerick, Louvain; Gauthier-Villars, Paris, 1957] a linear and non-negative (with respect to \mathcal{K}) mapping \mathcal{R} [resp. \mathcal{F}] from \mathcal{A} into \mathcal{A} is said to be a " \mathcal{R} -[resp. \mathcal{F} -] endomorphism" of Reynolds if it satisfies $2\mathcal{R}(f\mathcal{R}f) = (\mathcal{R}f)^2 + \mathcal{R}(\mathcal{R}f)^2$ [resp. $\mathcal{F}(f\mathcal{F}g) = \mathcal{F}f\mathcal{F}g$ and $\mathcal{F}e = e$] for every $f, g \in \mathcal{A}$, where e is the function in \mathcal{A} identically 1. Then every \mathcal{F} -endomorphism is an \mathcal{R} -endomorphism. In this paper the author assumes that E is a finite set. Whence \mathcal{A} is a finite dimensional vector space over the field \mathcal{K} and \mathcal{R} is represented as a finite matrix. After several lemmas the author proves theorem 1: \mathcal{R} can be expressed on its support as a matrix with a reducible form consisting of partial matrices P_j such that each P_j is characterized by the following three properties: (1) each element is positive or zero; (2) any two elements in the same column are equal to each other; (3) the sum of all elements in each row is equal to 1. As a corollary theorem 2 is proved: Every \mathcal{R} -endomorphism is a \mathcal{F} -endomorphism on its support.

[It may be pointed out that the \mathcal{F} -endomorphism in this paper has previously been treated in von Neumann algebra as a generalization of the concept of the conditional expectation [Dixmier, Bull. Soc. Math. France 81 (1953), 9-39; MR 15, 539; Nakamura and Turumaru, Tôhoku Math. J. (2) 6 (1954), 182-188; MR 16, 936; Umegaki, ibid. 6 (1954), 177-181; 8 (1956), 86-100; MR 16, 936; 19, 872; Tomiyama, Proc. Japan. Acad. 33 (1957), 608-612; MR 20#2635].] H. Umegaki (Tokyo)

5782:

Davis, Chandler. Compressions to finite-dimensional subspaces. Proc. Amer. Math. Soc. 9 (1958), 356-359.

In this paper formulae are derived for the coefficient c_r in

$$\det(\lambda + \{a_{ij}\}) = \sum c_r \lambda^{n-r},$$

where $PAPx_i = \sum a_{ij}x_j$ ($i=1, \dots, n$), A is a Hermitian operator in Hilbert space \mathcal{H} , P is a Hermitian projection, and x_1, \dots, x_n form a linear basis of $P\mathcal{H}$. The two key formulae are in terms of determinants defined as follows. G denotes the determinant of the $n \times n$ (Gramian) matrix with i, j entry (x_i, x_j) , i.e., $G = \det \{(x_i, x_j)\}$;

$$G(x_k; y_l) \cdots (x_{k_v}; y_l)$$

denotes the determinant of the matrix which differs from the Gramian in having k_i th row $(y_l, x_1), \dots, (y_l, x_n)$ for $i=1, \dots, v$.

The first formula is

$$c_r(PAP) = G^{-1} \sum G(x_k; Ax_k) \cdots (x_{k_v}; Ax_{k_v}).$$

The second formula is

$$c_r(PQP) =$$

$$(GH)^{-1} \sum G(x_k; y_l) \cdots (x_{k_v}; y_l) H(y_l; x_k) \cdots (y_l; x_{k_v}),$$

where Q is a Hermitian projection, y_1, \dots, y_m form a linear basis of $Q\mathcal{H}$, and $H = \det \{(y_l, y_j)\}$. The proof of the second formula is carried out in the notation of tensor products.

[It appears to the reviewer that without the restriction that A is Hermitian, the first formula is still valid and the appropriate generalization of the second formula is

$$c_r(PAP) =$$

$$(GH)^{-1} \sum G(x_k; y_l) \cdots (x_{k_v}; y_l) H(y_l; Ax_k) \cdots (y_l; Ax_{k_v}),$$

where y_1, \dots, y_m form a linear basis of the manifold spanned by Ax_1, \dots, Ax_n . Moreover, these formulae appear to be valid even when G and H have the more general form

$$G = \det \{(x_i, v_j)\}, \quad H = \det \{(y_l, w_j)\},$$

with v_j and w_j arbitrary provided $G \neq 0$, $H \neq 0$.

For a derivation of the above results (alternate to one contained in the paper) one simply evaluates

$$G(x_k; Ax_k) \cdots (x_{k_v}; Ax_{k_v})$$

after Ax_k ($i=1, \dots, v$) is replaced by its expansion in terms of either x_j (to obtain the first formula) or y_j (to obtain the second formula).]

H. Kurss (New York, N.Y.)

5783:

Lehrer, Yehiel. Sur la définition des fonctions de matrices. C. R. Acad. Sci. Paris 246 (1958), 870-872.

The author extends his definition of a function of a matrix [Rend. Circ. Mat. Palermo 6 (1957), 103-104; MR 20#1688] so as to apply to a many-valued function and give its different values. Essentially, the method is, if A has distinct latent roots $(\lambda_1, \dots, \lambda_n) = \lambda$, to take

$$f(A) = \frac{1}{v(\lambda)} (v_0(\lambda)I + v_1(\lambda)A + \cdots + v_{n-1}(\lambda)A^{n-1}),$$

where $v(\lambda) = |\lambda_j^{i-1}|$; the Vandermonde determinant, and $v_v(\lambda)$ is the determinant obtained from $v(\lambda)$ by substituting $f_{a_v}(\lambda_1), f_{a_v}(\lambda_2), \dots, f_{a_v}(\lambda_n)$ for its $(v+1)$ th row, where d_1, \dots, d_n indicate determinations (equal or different) of

the many-valued function $f(z)$. The case of equal latent roots is dealt with as a limiting case of the above.

H. S. A. Potter (Aberdeen)

5784:

Mirsky, L.; and Rado, R. A note on matrix polynomials. *Quart. J. Math. Oxford Ser. (2)* 8 (1957), 128-132.

Necessary and sufficient conditions are established to ensure that corresponding to a given polynomial $\phi(A)$ for a given matrix A there exists a polynomial $p(x)$ such that $p(\phi(A))=A$. The results are even extended to regular functions. Conditions which ensure that A is diagonalizable (or normal) together with $\phi(A)$ are obtained separately and it is pointed out that they also follow immediately from the earlier mentioned conditions. [For related work see also M. Mannos, *J. Research Nat. Bur. Standards* 51 (1953), 33-36; MR 15, 94].

O. Taussky-Todd (Pasadena, Calif.)

5785:

Sapiro, A. P. Characteristic polynomials of rational symmetric matrices of third order. *Dokl. Akad. Nauk SSSR (N.S.)* 119 (1958), 890-892. (Russian)

A purpose of this note is to prove the following theorem. In order that the irreducible polynomial $f(x)$ of third degree with rational coefficients be the characteristic polynomial of a rational symmetric matrix, it is necessary and sufficient that all its roots be real.

From the introduction

5786:

Krakowski, Fred. Eigenwerte und Minimalpolynome symmetrischer Matrizen in kommutativen Körpern. *Comment. Math. Helv.* 32 (1958), 224-240.

It is well known that symmetric matrices of real numbers have all their eigenvalues real. Here symmetric matrices with elements in an arbitrary field F are studied. It is shown that the eigenvalues of all symmetric matrices in F form a field Λ and that the eigenvalues of all symmetric matrices in Λ are again in Λ . If F is formally real then the eigenvalues of the symmetric matrices in F are exactly the elements which define totally real algebraic extensions of F . Otherwise any algebraic element over F can be an eigenvalue of a symmetric matrix. A zero of a quadratic polynomial $x^2+2ax+b$ (for characteristic $\neq 2$) is an eigenvalue of a symmetric F -matrix if a^2-b is a sum of squares in F . For characteristic 2 every square root is such an eigenvalue. In particular symmetric matrices over the rationals are discussed. Every real quadratic number is an eigenvalue of a 3×3 symmetric rational matrix and for a totally real algebraic number of degree n a matrix of order $\leq n \cdot 3^{n-1}$ will certainly do. {Reviewer's remark: for $n \times n$ rational symmetric matrices with eigenvalues of degree n see, e.g., D. K. Faddeev, *Dokl. Akad. Nauk. SSSR (N.S.)* 58 (1947), 753-754 [MR 9, 270] and O. Taussky-Todd, *Illinois J. Math.* 1 (1957), 108-113 [MR 20 #845]. In particular \sqrt{m} can be an eigenvalue of a 2×2 rational matrix only if m is a sum of two squares.} Analogous theorems are proved for the minimal polynomials of symmetric matrices; here characteristic 2 plays an exceptional role. Finally skew symmetric and orthogonal matrices are discussed.

O. Taussky-Todd (Pasadena, Calif.)

5787:

Marathe, C. R. On certain moduli of rectangular matrices. *Proc. Roy. Soc. Edinburgh. Sect. A.* 65 (1957), 13-28.

If $A=(a_{ik})$ is a rectangular matrix over the complex field it is shown that $R(A)=\max_i \sum |a_{ik}|$, $C(A)=\max_k \sum |a_{ik}|$ and $|A|^0$, the square root of the maximum

characteristic root of A^*A , are moduli in the sense of Y. K. Wong [*Proc. Amer. Math. Soc.* 6 (1955), 891-899; MR 17, 936]. Some properties of $R(A)$, $C(A)$, $|A|^0$ and the corresponding quantities obtained by taking minima instead of maxima are discussed; they are applied to obtain estimates of bounds on the characteristic roots of sums and products of matrices.

O. Taussky-Todd (Pasadena, Calif.)

5788:

Fan, Ky. Note on circular disks containing the eigenvalues of a matrix. *Duke Math. J.* 25 (1958), 441-445.

The main result concerns the following problem: Consider all square $n \times n$ matrices $A=(a_{ij})$ with complex elements such that $|a_{ij}|$ are given fixed numbers for $i \neq j$. Determine all ordered n -tuples of positive numbers ρ_1, \dots, ρ_n such that every eigenvalue of A lies inside or on the boundary of the n circles with centres a_{ii} and radii ρ_i . This question is answered for indecomposable A 's only, but this is not an essential restriction. The answer is: all sets ρ_i for which there exist n positive numbers x_1, \dots, x_n such that $\rho_i \geq (1/x_i) \sum_{j \neq i} |a_{ij}| x_j$. The proper interpretation of the result is probably obtained by identifying these last expressions with the Gersgorin radii corresponding to the matrix $X^{-1}AX$ where X is the diagonal matrix (x_1, \dots, x_n) . The paper contains a number of other remarks concerning eigenvalues of finite matrices.

O. Taussky-Todd (Pasadena, Calif.)

5789:

Schneider, Hans. Note on the fundamental theorem on irreducible non-negative matrices. *Proc. Edinburgh Math. Soc.* 11 (1958/59), 127-130.

This paper is concerned with a difficulty in a proof of the Frobenius theorems as given by Wielandt [*Math. Z.* 52 (1950), 642-648; MR 11, 710]. To overcome the difficulty the author obtains the following lemma: If $y > 0$, $Ay \leq My$, $y_1 \geq y_2 \geq \dots \geq y_n$; and if κ is a least diagonal element of A , λ a least nonvanishing nondiagonal element of A , then

$$y_n/y_1 \geq \lambda^{n-1}(M-\kappa)^{-(n-1)}.$$

In particular, if $Ax = \rho x > 0$, then

$$\min x_i / \max x_i \geq \lambda^{n-1}(\rho - \kappa)^{-(n-1)}.$$

A. S. Householder (Oak Ridge, Tenn.)

5790:

Lotze, Alfred. Die projektive Invariantentheorie von Polarsystemen und ihren Kerngebilden im Lichte der Grassmannschen Punktrechnung. *Jber. Deutsch. Math. Verein.* 60 (1957), Abt. 1, 77-89.

L'Auteur présente la théorie des invariants des systèmes polaires en utilisant de manière systématique le calcul géométrique de Grassmann qui permet un exposé élégant.

G. Papy (Brussels)

5791:

Kremneva, Yu. P. Nonhomogeneous bilinear systems. *Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr.* 12 (1957), no. 19, 79-86. (Russian. English summary)

"In this paper we consider the bilinear system

$$\sum_{i=0}^n \sum_{j=0}^m a_{ij} x_i y_j = b_s \quad (s=0, 1, \dots, (m+n))$$

consisting of $(m+n+1)$ equations. Coefficients a_{ij} , b_s belong to the field of all complex numbers. If the determinant Δ , consisting of the coefficients of the system, differs from zero, we have: 1) every $(x_i; y_j)$ is the root of the characteristic equation of the matrices of special type

B_{ij} of order $(m+n)!/n!m!$; 2) the ring of the matrices $\Omega(B_{ij})$ is commutative; 3) the number of solutions of the bilinear system $\leq (m+n)!/n!m!$. *Author's summary*

ASSOCIATIVE RINGS AND ALGEBRAS

See also 5781, 5831.

5792:

Mrówka, S. A remark concerning the multiplicative linear functionals. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 6 (1958), 309-311.

This paper contains sharpened versions of theorems of Białynicki-Birula and Zelazko [*Bull. Acad. Polon. Sci., Cl. III* 5 (1957), 589-593; MR 19, 526] on the form of a multiplicative linear functional on a Cartesian product of algebras. The sharpened theorems apply, for example, to algebras over a finite field. The proofs are substantially the same as those of the original theorems.

M. Jerison (Princeton, N.J.)

5793:

Hoehnke, Hans-Jürgen. Über komponierbare Formen und konkordante hyperkomplexe Größen. *Math. Z.* 70 (1958), 1-12.

Let \mathfrak{A} be a finite-dimensional associative algebra (over a field) with a unit element e . A general element ξ of \mathfrak{A} is defined to be a linear combination of basis elements with indeterminate coefficients x_1, \dots, x_m . A composition form $G(\xi)$ is a form in the x_i 's satisfying $G(\xi\eta) = G(\xi)G(\eta)$ for any two general elements ξ, η . For a composition form $G(\xi)$, define the forms $g_1(\xi), g_2(\xi), \dots$ by

$$G(\lambda\xi - \xi) = \lambda^n - g_1(\xi)\lambda^{n-1} + \dots + (-1)^n g_n(\xi).$$

The author derives many relations between $g_k(\xi)$, $g_k(\xi)$, etc., and uses these to solve the following "concordance problem": If ξ and η are two general elements, then $g_k(\xi\eta)$ can be computed solely from the forms $g_i(\xi)$.

R. Ree (Vancouver, B.C.)

5794:

Fadini, Angelo. Composizione delle algebre. *Giorn. Mat. Battaglini* (5) 5(85) (1957), 172-187.

L'auteur expose quelques résultats élémentaires relatifs à la théorie des algèbres finies dont les coefficients appartiennent à une algèbre finie sur un champ.

G. Papy (Brussels)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

5795:

Paige, Lowell J. A note on noncommutative Jordan algebras. *Portugal. Math.* 16 (1957), 15-18.

Let L be an algebra over a field F of characteristic $\neq 2$, such that $x^2=0$ (all x in L) and such that there is a bilinear form $f(x, y)$ with values in F satisfying

$$f(x, y) = f(y, x), \quad f(xy, z) = f(x, yz) \quad (\text{all } x, y, z \text{ in } L).$$

With t, s in F , an algebra $A = A(L, f, t, s)$ is constructed as follows. A consists of matrices

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \quad (a, b \text{ in } F; \alpha, \beta \text{ in } L);$$

addition and scalar multiplication of such matrices are

defined in the obvious way, and multiplication by the rule

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} = \begin{pmatrix} ac + f(\alpha, \delta) & a\gamma + ad + t\beta\delta \\ c\beta + b\delta + s\alpha\gamma & f(\beta, \gamma) + bd \end{pmatrix}.$$

Then A is flexible, of degree 2, and hence is a non-commutative Jordan algebra. A is simple if and only if the form f is non-degenerate on L . In particular, if L is a semisimple Lie algebra of characteristic 0 and f is its Killing form, A is simple.

I. M. H. Etherington (Edinburgh)

5796:

Širšov, A. I. On free Lie rings. *Mat. Sb. N.S.* 45 (87) (1958), 113-122. (Russian)

If Σ is an associative ring with identity and if R is a set of symbols, let $\mathfrak{A}_{\Sigma R}$ denote the free Σ -operator associative ring with the set R of free generators and let $\mathfrak{A}_{\Sigma R}^{(-)}$ denote the Lie subring which R generates in $\mathfrak{A}_{\Sigma R}$ (with the operation $xoy = xy - yx$). The ring $\mathfrak{A}_{\Sigma R}^{(-)}$ and the free Σ -operator Lie ring with R as its set of free generators are isomorphic [Širšov, *Uspehi Mat. Nauk* (N.S.) 8 (1953), no. 5(75), 173-175; MR 15, 596]. A criterion for deciding whether a given element of the ring $\mathfrak{A}_{\Sigma R}^{(-)}$ belongs to the Lie ring $\mathfrak{A}_{\Sigma R}^{(-)}$ is investigated [K. O. Friedrichs, *Comm. Pure Appl. Math.* 6 (1953), 1-72; MR 15, 80]. Every Lie algebra of at most countable rank is isomorphically embeddable in a Lie algebra with two generators over the same field. The corresponding results for restricted Lie rings are also given. *R. A. Good (College Park, Md.)*

5797:

Burrow, Martin D. Invariants of free Lie rings. *Comm. Pure Appl. Math.* 11 (1958), 419-431.

Every (unimodular) invariant of degree d in a free Lie ring on n generators can be gotten from an invariant form J in a related associative ring R' on nd generators. The Lie invariant can then be written $\Omega_d J$, where $\Omega_d = (1 - \Pi_d) \dots (1 - \Pi_2)$ with Π_k the cyclic permutation $(12 \dots k)(k+1) \dots (n)$. Knowing the associative invariants (products of determinants of order n) the Lie invariants can thus be found by determining those J for which $\Omega_d J \neq 0$. For $n=2$ Wever [*Math. Ann.* 120 (1949), 563-580; MR 10, 676] combined this approach with character theory to get formulas for the number of invariants of each degree. He also showed that there were no invariants of degree $d=n$ for any $n \geq 3$, and none of degree 6 when $n=3$.

In the paper under review the author constructs J 's having $\Omega_d J \neq 0$ in all the remaining possible degrees: all $d=mn$ with $n \geq 4$ and $m \geq 2$. He gives a general rule to pick out one J of each degree, and argues in various separate cases to show that the coefficient of a certain "leading term" of $\Omega_d J$ is in each case non-zero.

R. L. Davis (Charlottesville, Va.)

5798:

Tomber, Marvin L. Lie algebras of types A, B, C, D, and F. *Trans. Amer. Math. Soc.* 88 (1958), 99-106.

By finding common representations in associative algebras Jacobson [*Ann. of Math.* 50 (1949), 866-874; MR 11, 76] showed that the simple Lie algebras of characteristic 0 in the infinite classes are derivation algebras of central simple Jordan algebras. In this paper Lie algebra methods used by the author to get a Jordan derivation representation of the Lie algebras of type F [*Proc. Amer. Math. Soc.* 4 (1953), 759-768; MR 15, 195] are employed to establish theorems which imply some of Jacobson's. The critical step is the proof that, generally, if central simple Jordan algebras have isomorphic derivation

algebras, the isomorphism can be uniquely realized as one induced by an isomorphism of the algebras themselves.

W. G. Lister (Oyster Bay, N.Y.)

5799:

Kaplansky, Irving. Lie algebras of characteristic p . Trans. Amer. Math. Soc. 89 (1958), 149-183.

The author attacks here, with considerable success, the problem of classifying all simple Lie algebras, over an arbitrary algebraically closed field, possessing Cartan subalgebras of dimensions 1 and 2. In the classical case of characteristic zero, as well as in more recent work in prime characteristic, it may be noted that by proving (or assuming) a few properties of subalgebras of these types in simple Lie algebras, as well as that a (commutative) Cartan subalgebra acts diagonally, machinery can be set up for a complete analysis of the simple algebras [see, e.g., Mills and Seligman, J. Math. Mech. 6 (1957), 519-548; MR 19, 631]. The cases of characteristics 2 and 3 do not fall in the domain of the principal techniques, and are treated in an appendix; nothing more about them will be said here. There is also one point of difficulty in characteristic 5; the author has added in proof that he has eliminated this difficulty, and he has also announced at the end of the paper some extensions of the results of the paper for characteristic at least 5. Salient among the latter are: Theorem 3: If L is a restricted simple Lie algebra of rank one, then L is either three-dimensional or is the Witt algebra; Theorem 5: If L is a Lie algebra of rank one possessing a non-singular invariant bilinear form (i.b.f.), then L is 3-dimensional; Theorem 7: If L is restricted, simple, of rank two, and possesses a nonsingular i.b.f., then any two-dimensional Cartan subalgebra of L acts diagonally; Theorem 6: If L is a Lie algebra with center zero, with a non-singular i.b.f. and a Cartan subalgebra that acts diagonally, then all roots relative to this Cartan subalgebra are non-isotropic. It is remarked, in effect, that simple restricted Lie algebras satisfying the hypotheses of Theorem 6 fall in the category of those algebras axiomatized and classified by W. H. Mills and the reviewer in the article cited above. Combining this remark with Theorem 7, we see that the only Lie algebras satisfying the hypotheses of Theorem 7 are those of the classical types A_3 , B_3 (or C_3), and G_2 . The results are obtained by building up long chains of lemmas, each of a simple but technical nature. The hope that they can help to bring order out of the general chaos in characteristic p is somewhat dampened by the author's remark that algebras of rank three can display pathologies not anticipated from the results for ranks one and two.

G. B. Seligman (Muenster)

HOMOLOGICAL ALGEBRA

See also 6092.

5800:

Matlis, Eben. Injective modules over Noetherian rings. Pacific J. Math. 8 (1958), 511-528.

This paper is chiefly concerned with indecomposable, injective modules over a commutative, Noetherian ring R . First of all, it is shown that by associating with each prime ideal P of R the injective envelope $E = E(R/P)$ of R/P in the sense of Eckmann-Schopf [Arch. Math. 4 (1953), 75-78; MR 15, 5] we have a one-to-one correspondence

between the prime ideals of R and the indecomposable, injective R -modules. For any non-zero element x of E there exists a positive integer i such that $P^i x = 0$, and each element of R which is not in P operates on E as an automorphism. These enable us to define E as a module over the P -adic completion R_P of R in a natural way, and we have the following main theorem: E is the indecomposable, injective R_P -module corresponding to the unique maximal ideal of R_P , and moreover R_P coincides with the R -endomorphism ring of E . This is apparently a generalization of the well-known fact that if R is the ring of rational integers, p a prime number, and E the p -component of the additive group of rational numbers reduced mod 1, then the endomorphism ring of E is the ring R_P of p -adic integers. A duality theorem for a complete, Noetherian, local ring R is deduced from the above theorem: there is a one-to-one correspondence between R -modules A having the ascending chain condition and R -modules D having the descending chain condition, by the following dual relations: $\text{Hom}_R(A, E) = D$, $\text{Hom}_R(D, E) = A$.

The paper also contains some results on injective modules over a noncommutative ring R . For example, if R is left-Noetherian, then every left R -module has a maximal injective submodule, which is unique up to automorphism, and every injective left R -module is decomposed into a direct sum of indecomposable, injective submodules, which are also unique up to automorphisms, while a ring R is left-hereditary if and only if every left R -module has a (unique maximal) injective submodule which contains all injective submodules.

G. Azumaya (Evanston, Ill.)

GROUPS AND GENERALIZATIONS

See also 5751, 6090.

5801:

*Alexandrov, P. S. An introduction to the theory of groups. Translated by Hazel Perfect and G. M. Petersen. Hafner Publishing Co., Inc., New York, 1959. viii+112 pp. \$3.25.

A translation from the German [Deutscher Verlag der Wissenschaften, Berlin, 1954] reviewed in MR 16, 791. A Polish translation [Państwowe Wydawnictwo Naukowe, Warsaw, 1956] is listed in MR 18, 279. The original Russian, was published by Utcpedgiz, Moscow, 1955.

5802:

Mostowski, A. Włodzimierz. On direct sums of cyclic groups. Prace Mat. 2 (1958), 319-328. (Polish. Russian and English summaries)

J. Szele [Publ. Math. Debrecen 2 (1951), 76-78; MR 13, 104] has given a condition of extremality of a set Z of generators of an abelian group G that is sufficient in order that the set Z be a basis in the group. The condition is not necessary. In the direct sum $C_{p_1} \times C_{p_2} \times C_{p_3} \times \dots$, where C_{p_i} are cyclic groups of prime orders, $p_1 < p_2 < \dots$, no set of generators is extremal.

The modified condition of Szele gives the following: Theorem: The set Z of generators, different from zero, of an abelian group G is a basis of G if and only if for arbitrary finite sets X and Y , generating the same subgroup of the group G , we have the implications $XCN \cap Z$ and YCG imply the power of $X \leq$ the power of $N \cap Y$, $XCS \cap Z$ and $YCS \cap G$ imply $\prod_{x \in X} r(x) \leq \prod_{y \in Y} r(y)$. N and S

denote sets of elements of infinite and of finite orders of the group G . The number $\tau(x)$ ($1 \leq \tau(x) \leq \infty$) denotes the order of the element x .

This theorem implies in a simple way the theorem for finitely generated abelian groups. *Author's summary*

5803:

Cassels, J. W. S. On the subgroups of infinite Abelian groups. *J. London Math. Soc.* 33 (1958), 281-284.

Let A be an additively written torsion-free abelian group; let a_1, \dots, a_s be elements none of which can be written in the form nb (n is an integer > 1 , $b \in A$); let j_1, \dots, j_s be positive integers and put $J = j_1 + \dots + j_s$. Then there exists a subgroup B of A of index

$$(1) \quad [A:B] \leq J+1$$

which contains none of the J elements $i_\sigma a_\sigma$ ($1 \leq i_\sigma \leq j_\sigma$; $\sigma = 1, \dots, s$). There can be strict inequality in (1) except when $s=1$. In the special case when A is a free abelian group of finite rank, this theorem yields a conjecture of C. A. Rogers [*J. London Math. Soc.* 26 (1951), 307-310; MR 14, 624] which was partially proved earlier by Rogers and by W. Schmidt [*Monatsh. Math.* 59 (1955), 274-304; MR 17, 1188].

A. Kertész (Debrecen)

5804:

Solian, Aleksandru. Sur la notion de " n -complet" dans les groupes. *Ž. Čist. Prikl. Mat.* 1 (1956), 1-22. (Russian)

Translated from the French [*Acad. R. P. Romine. Bul. Ști. Secț. Ști. Mat. Fiz.* 7 (1955), 255-272; MR 17, 1183].

5805:

Berman, S. D.; and Bovdi, A. A. p -blocks for one class of finite groups. *Dopovidi Akad. Nauk Ukraïn. RSR* 1958, 606-608. (Ukrainian. Russian and English summaries)

Let G be a group of order $p^\alpha q$, where p is a prime number and $(p, q)=1$. Let G contain a normal subgroup H of order $p^\gamma q$ ($0 \leq \gamma \leq \alpha$) the p -Sylow subgroup of which is a normal divisor of the subgroup in H .

Let T be the maximal normal subgroup of G with order $m \not\equiv 0 \pmod{p}$. Let $R(G, K)$ denote the group algebra of group G over the arbitrary field K of characteristic p .

Then the number of indecomposable constituents in the direct two-sided decomposition of the algebra $R(G, K)$ is equal to the number of classes of K -conjugate elements of group G which lie in T . Two irreducible ordinary characters χ_i and χ_j belong to the same block if and only if χ_i and χ_j induce over T the same character. *Authors' summary*

5806:

Deskins, W. E. A note on the relationship between certain subgroups of a finite group. *Proc. Amer. Math. Soc.* 9 (1958), 655-660.

It is known that if H is a normal subgroup of the finite group G , then an irreducible G -module (over any field F) either remains irreducible as an H -module or decomposes into a direct sum of conjugate irreducible H -modules. Assuming this decomposition property with respect to a field F (taken to be large enough to make irreducible representations absolutely irreducible) the author asks as to the relationship of the subgroup H to the group G . First it is shown that a subgroup K exists with $H \triangleleft K \triangleleft G$ and that H has property I in K , relative to the field F . Property I says that each irreducible K -module remains

irreducible as an H -module. Property I is shown to be equivalent to the following: if H has s H -conjugate classes and if $[K:H]=n$, then K has ns K -conjugate classes, the classes in H remaining classes in K . Using a result of the reviewer [*Ann. of Math.* 38 (1938), 220-234] it is shown that under certain circumstances K must be the direct product of H and an Abelian group. Specifically, if $[H:1]=h$ and if $(h, n)=1$, and the characteristic of F does not divide $g=hn$, this conclusion follows. Various other situations are investigated.

Marshall Hall, Jr. (Columbus, Ohio)

5807:

Edge, W. L. The partitioning of an orthogonal group in six variables. *Proc. Roy. Soc. London. Ser. A.* 247 (1958), 539-549.

This paper studies the projective orthogonal group G in six variables over $GF(3)$. The projective space contains 364 points of which 112 lie on the quadric $Q=x_1^2+\dots+x_6^2=0$, and 126 have $Q=+1$, and 126 have $Q=-1$. These points are designated generically as m, k, l . This is an extension of two earlier papers [same Proc. 233 (1955), 126-146; *Proc. London Math. Soc.* (3) 8 (1958), 416-446; MR 17, 941; 20 #3853].

The conjugate classes of G are partitioned into types depending on the cycle patterns of the points of the geometry distinguishing points of type m, k, l . This is done with relative ease using the earlier paper on the similar group in five variables.

Marshall Hall, Jr. (Columbus, Ohio)

5808:

Conrad, Paul. A note on valued linear spaces. *Proc. Amer. Math. Soc.* 9 (1958), 646-647.

The method used by Banaschewski for a proof of Hahn's embedding theorem [*Arch. Math.* 7 (1956), 430-440; MR 19, 385] is applied in order to give a simple proof of a generalization of Hahn's theorem that has been proved recently by Gravett [*Quart. J. Math. Oxford Ser. (2)* 6 (1955), 309-315; MR 19, 385]. The author notes that the same method is applicable to a more general theorem found by himself [*Amer. J. Math.* 75 (1953), 1-29; MR 14, 842].

L. Fuchs (Budapest)

5809:

Choe, Tae-Ho. Notes on the lattice-ordered groups. *Kyungpook Math. J.* 1 (1958), 37-42.

Let L be a lattice-ordered group, and let n be a positive integer. The third theorem of this paper asserts that if there exist distinct elements x_1, \dots, x_n of L such that each x_i covers 0, and such that if $y > 0$ then $y \geq x_i$ for some i , then L is the cardinal product of n infinite cyclic groups. This is false: the cardinal product of n discrete linearly ordered groups satisfies the hypothesis. But the theorem is true if it is also assumed that L is archimedean. Theorems 1 and 2 are special cases of the above for $n=1$ and 2, and they are also false as stated. The last part of the paper contains some results about the open interval topology on L . The proofs are long and straightforward, but all the results are immediate consequences of the "well-known" fact that if L is not linearly ordered, then every element of L is the intersection of two open intervals.

P. F. Conrad (New Orleans, La.)

5810:

Šik, František. Über Summen einfach geordneter Gruppen. *Czechoslovak Math. J.* 8(83) (1958), 22-53. (Russian summary)

The author considers lattice ordered groups (l -groups)

G that are complete direct sums, subdirect sums, or completely subdirect sums of simply ordered groups.

First, an element x of an l -group G is defined to be a vertex of an element a , $0 < a \in G$, if x is a minimal element with the property $a \geq x > 0$, $|a - x| \wedge |x| = 0$. It is shown that an l -group G is simply ordered if and only if every positive element of G is the vertex of some element a , $0 < a \in G$. Thereafter, the following theorem is typical. "For an l -group $G \neq 0$ the following conditions are equivalent. (1) G is a completely subdirect sum of a system of simply ordered groups. (2) In G the following hold: (a) Each element $a \in G$, $a > 0$, has a vertex; (b) if $x \geq y > 0$ and x is a vertex, y is a vertex; (c) if y is a vertex, $a \geq y$, then there is a vertex of the element a greater than y . (3) For G there exists a system of pairwise disjoint, simply ordered direct factors that are complete in G . (4) Every component different from zero of G contains a minimal direct factor of G ." L. J. Paige (Los Angeles, Calif.)

5811:

Nishigōri, Noboru. On some properties of FC -groups. J. Sci. Hiroshima Univ. Ser. A 21 (1957/58), 99-105.

An FC -group is a group in which each class of conjugates is a finite set [R. Baer, Duke Math. J. 15 (1948), 1021-1032; MR 10, 352]. If K is a subgroup of an FC -group G , then the author shows that the division hull of K in G , the set of all x in G with some power in K , is a subgroup of G . Further, he proves that simple FC -groups are finite groups; and finitely generated FC -groups turn out to be direct products of a finite number of copies of the integers and of a finite group. This last result is a somewhat sharper form of the result of B. H. Neumann [Proc. London Math. Soc. (3) 1 (1951), 178-187; MR 13, 316] that every finitely generated FC -group is an extension of a free abelian group on a finite number of generators by a finite group. F. Haimo (St. Louis, Mo.)

5812:

Erugin, I. I. Groups with finite classes of conjugate Abelian subgroups. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 223-224. (Russian)

If every abelian subgroup of the group G has only a finite number of conjugates, then G is a finite extension of its center. Neumann had proved this theorem in weaker form ("abelian" omitted) [Math. Z. 63 (1955), 76-96; MR 17, 234]. J. L. Brenner (Menlo Park, Calif.)

5813:

Shiratani, Katsumi. On the characters and the character rings of finite groups. Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math. 11 (1957), 99-115.

Sei \mathcal{G} eine endliche Gruppe und p eine Primzahl. Eine Untergruppe \mathcal{E} von \mathcal{G} heißt p -elementar, wenn sie direktes Produkt $\mathcal{E} = (A) \times \mathcal{B}$ einer p -Gruppe \mathcal{B} mit einer zyklischen Gruppe (A) von zu p teilerfremder Ordnung a ist. Eine Aufgabe der Darstellungstheorie ist die Untersuchung des Zusammenhanges der Charaktere von \mathcal{G} mit den Charakteren der p -elementaren Untergruppen von \mathcal{G} . [Vgl. z.B. Brauer und Tate, Ann. of Math. (2) 62 (1955), 1-7; MR 16, 1087.] Im Hinblick hierauf gibt Verf. hier ein notwendiges und hinreichendes Kriterium dafür, daß eine komplexwertige Klassenfunktion θ auf \mathcal{G} ein "verallgemeinerter" Charakter von \mathcal{G} (d.h. ein Element der von den Charakteren erzeugten additiven Gruppe) ist; und zwar bezieht sich dieses Kriterium auf das Verhalten von θ auf den p -elementaren Untergruppen \mathcal{E} von \mathcal{G} . Sei $\theta(\mathcal{E})$ diejenige Funktion auf $\mathcal{E} = (A) \times \mathcal{B}$,

welche für die Elemente der Form $A^b \cdot P$, $(b, a) = 1$, $P \in \mathcal{B}$ mit θ übereinstimmt und sonst 0 ist. Das erwähnte Kriterium des Verf. besagt dann: Zu jeder Primzahl p soll es eine ganzrationale, zu p teilerfremde Zahl l_p geben derart, daß $l_p \cdot \theta(\mathcal{E})$ für jede p -elementare Untergruppe \mathcal{E} ein verallgemeinerter Charakter von \mathcal{E} ist. Dieses Kriterium ist verwandt, jedoch nicht identisch mit dem entsprechenden Kriterium von R. Brauer [vgl. hierzu die oben zitierte Arbeit von Brauer und Tate]. — Als Hilfsmittel zum Beweis gibt Verf. u.a. eine Charakterisierung der p -elementaren Untergruppen von \mathcal{G} durch die Werte, die die induzierten Charaktere auf ihnen annehmen [S. 113, Theorem 8; vgl. dazu auch J. A. Green, Proc. Cambridge Philos. Soc. 51 (1955), 237-239; MR 16, 565]. Ferner wird auch die Struktur des p -adischen Charakteres $X_p(\mathcal{G})$ in Bezug auf eine ganzrationale Primzahl p untersucht [§ 5, S. 110-113; hierzu vgl. auch S. D. Berman, Dokl. Akad. Nauk. SSSR 106 (1956), 583-586; MR 17, 1052]. P. Roquette (Hamburg)

5814:

Preston, G. B. Matrix representations of semigroups. Quart. J. Math. Oxford Ser. (2) 9 (1958), 169-176.

Let S be a semigroup with identity. The \mathcal{R} - and \mathcal{L} -classes of S are defined by the equivalences $a \mathcal{R} b \Leftrightarrow \{aS = bS\}$, $a \mathcal{L} b \Leftrightarrow \{Sa = Sb\}$. The \mathcal{D} -classes of S are defined by means of the "product" $\mathcal{L} \cdot \mathcal{R} = \mathcal{R} \cdot \mathcal{L}$ as introduced by Green [Ann. of Math. (2) 54 (1951), 163-172; MR 13, 100].

To any \mathcal{D} -class of S we may assign uniquely (up to an isomorphism) a group G with zero. Corresponding to each \mathcal{D} -class we may then construct two matrix representations $s \rightarrow M(s)$, $s \rightarrow M'(s)$ ($s \in S$) of S with entries from G (the matrices being multiplied in an obvious way). This is a generalization of a result of Schützenberger [C. R. Acad. Sci. Paris 244 (1957), 1994-1996; MR 19, 249], who proved the same statements for \mathcal{D} -classes of special type ("type élémentaire"). The methods of proofs are essentially the same as those of Schützenberger.

Denote by $\mathcal{D}(s)$ [respectively $\mathcal{D}'(s)$] the direct sum of all representations $s \rightarrow M(s)$ [respectively $s \rightarrow M'(s)$] corresponding to each of the \mathcal{D} -classes of S . Let $[\mathcal{D} + \mathcal{D}'](s)$ denote the direct sum of $\mathcal{D}(s)$ and $\mathcal{D}'(s)$. Necessary and sufficient conditions are given for each of the representations $\mathcal{D}(s)$, $\mathcal{D}'(s)$, $[\mathcal{D} + \mathcal{D}'](s)$ to be faithful. In particular, (1) if S is a regular semigroup (in the sense of Green, l.c. above), then $[\mathcal{D} + \mathcal{D}'](s)$ is a faithful representation of S ; (2) if S is an inverse semigroup, $\mathcal{D}(s)$ and $\mathcal{D}'(s)$ are each faithful representations.

An example of a semigroup which is not regular but which is faithfully represented by $\mathcal{D}(s)$ is given.

S. Schwarz (Bratislava)

5815:

Frigerio, Alberto. Sui quasi gruppi associati ai gruppi. Rend. Sem. Mat. Univ. Padova 28 (1958), 107-111.

Given a group G , the operation $aOb = ab^{-1}$ defines a quasigroup Q called the associate of G . A quasigroup Q is the associate of some group G if and only if (i) Q contains an element u with the property $aOb = u$ if and only if $a = b$, (ii) $(aOc)O(bOc) = aOb$ for all a, b, c of Q ; G is then determined by $ab = aO(bOb)$. There is a one-one correspondence between the subgroups of G and the subquasigroups of Q , which are their associates; and there is an isomorphism between the lattices of congruence relations on G and on Q . When G is finite, a theorem of Jordan-Hölder type holds for Q .

(It may be remarked that in the terminology of S. K.

Stein [Trans. Amer. Math. Soc. 85 (1957), 228-256; MR 20 #922] Q is the (13) conjugate of G . Condition (i) above is deducible from (ii); cf. Stein's Corollary 5.6.

I. M. H. Etherington (Edinburgh)

5816:

Etherington, I. M. H. Groupoids with additive endomorphisms. Amer. Math. Monthly 65 (1958), 596-601.

If G is a groupoid and $f, g: G \rightarrow G$ are functions, then $f+g$ is defined by $(f+g)x = f(x)g(x)$ and $f \cdot g$ is defined by $(f \cdot g)x = g(f(x))$. Every word W in the symbol x (such as x or $(xx)x$) defines $\psi(W): G \rightarrow G$. If U, V are words in x , then the word formed by replacing x in V by U shall be denoted UV . Note that $\psi(UV) = \psi(U) \cdot \psi(V)$.

The set of all functions $f: G \rightarrow G$ of the form $\psi(W)$ is (essentially) the "logarithmic" L of G . Clearly, if $f, g \in L$ so are $f+g$ and $f \cdot g$.

The primary concern of the paper is the interplay of G and L . For example: if the sum $f+g$ of any two endomorphisms of G is an endomorphism, then $\psi(UV) = \psi(VU)$ for any two words U and V ; also, every element of L is an endomorphism of G ; if, furthermore, G is generated by a single element, then L is precisely the set of endomorphisms of G .

S. Stein (Davis, Calif.)

5817:

Robinson, Donald W. n -groups with identity elements. Math. Mag. 31 (1957/58), 255-258.

A n -group is an algebraic system with one $(n+1)$ -ary associative operation for which the equations

$$xa_1 \cdots a_n = a, a_1 \cdots a_n y = a$$

are always solvable for x and y . It is shown that an algebraic system with an $(n+1)$ -ary associative operation and an element e such that $e \cdots ea = a$ for every a and $xe \cdots ea = e$ is always solvable for x is a n -group, and that then $ae \cdots e = a$ for every a and $ae \cdots ey = e$ is always solvable for y . A n -group may have more than one element with these properties.

H. A. Thurston (Vancouver, B.C.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 5842, 6101.

5818:

Williamson, J. H. On theorems of Kawada and Wendel. Proc. Edinburgh Math. Soc. 11 (1958/59), 71-77.

Let G be a locally compact group with left invariant Haar measure; let L denote the usual B -algebra of integrable functions. A bounded Borel measure m determines a linear map of L to itself by means of the correspondence $f \rightarrow m * f$, where $*$ denotes convolution. If the map is onto, and if $f \geq 0 \Leftrightarrow m * f \geq 0$, then a theorem of Kawada [Math. Japon. 1 (1948), 1-5; MR 10, 11] shows that m is a point-mass. Clearing up an ambiguity left by the reviewer [Pacific J. Math. 2 (1952), 251-261; MR 14, 246] the author constructs an example to show that the result fails, in general, if the condition "onto" is replaced by "into". He goes on to show that the "into" theorem is still valid in a range of cases, including those in which G is abelian, discrete, or compact, and conjectures that unimodularity is necessary and sufficient. His methods apparently furnish simpler proofs of results of the reviewer's [loc. cit.].

J. G. Wendel (Ann Arbor, Mich.)

5819:

Iwahori, Nobuko. A proof of Tannaka duality theorem. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 8 (1958), 1-4.

The author gives a simple proof of the well-known Tannaka duality theorem in the following form. Let G be a compact group and $R(G)$ the ring of Fourier polynomials on G . Let T be a linear map of $R(G)$ into $R(G)$ which commutes with all left-translations by elements of G and satisfies (a) $T1 = 1$, (b) $\overline{Tf} = T\overline{f}$, and (c) $T(fg) = Tf \cdot Tg$ for $f, g \in R(G)$. Then T coincides with a right translation R_a for some $a \in G$. An essential point in the proof is to give a simple deduction of the continuity of T with respect to the uniform norm from the given algebraic conditions on T .

F. I. Mautner (Paris)

5820:

Dixmier, J. Sur les représentations unitaires des groupes de Lie algébriques. Ann. Inst. Fourier, Grenoble 7 (1957), 315-328.

Let G be a topological group and U a unitary representation of G . If the ring of operators A in the sense of von Neumann generated by the operators $U(g)$ is of type I, then U is called of type I. The problem of determining the relationship between the structure of G and the possible types of its unitary representations is far from being solved. The present paper settles the question for real algebraic groups. Indeed, it is shown (Theorem 1) that every unitary representation of (the connected component of) any real algebraic group is of type I. The proof uses Harish-Chandra's result [Trans. Amer. Math. Soc. 75 (1953), 185-243; MR 15, 100] that every semi-simple Lie group is of type I and the present author's result that the same is true for nilpotent Lie groups [to appear J. Math. Pures Appl.]. Use is also made of a result of Chevalley [not yet published] about the behavior of the orbits of an algebraic transformation group. The proof proceeds through a study of certain nilpotent Lie algebras toward showing that every connected reductive Lie group is of type I. In order to prove the theorem now for arbitrary algebraic groups G , use is made of the existence of an ideal of nilpotent endomorphisms in the Lie algebra of G whose quotient algebra is reductive. It is now necessary to study this particular group-extension in detail and to show that the property of being of type I is preserved under this particular kind of group-extension.

F. I. Mautner (Paris)

5821:

Tsou, S-T.; and Walker, A. G. Metrisable Lie groups and algebras. Proc. Roy. Soc. Edinburgh. Sect. A. 64 (1955-57), 290-304.

"A Lie group is said to be metrisable if it admits a Riemannian metric which is invariant under all translations of the group. It is shown that the study of such groups reduces to the study of what are called metrisable Lie algebras, and some necessary conditions for a Lie algebra to be metrisable are given. Various decomposition and existence theorems are also given, and it is shown that every metrisable algebra is the product of an abelian algebra and a number of non-decomposable reduced algebras. The number of independent metrics admitted by a metrisable algebra is examined, and it is shown that the metric is unique when and only when the complex extension of the algebra is simple." (Authors' summary).

L. Auslander (Bloomington, Ind.)

5822:

Ruse, H. S. On the geometry of metrisable Lie algebras. Proc. Roy. Soc. Edinburgh. Sect. A. 65 (1957), 1-12.

The object of this paper is to display something of the geometrical background of metrisable Lie algebras as defined by Tsou and the reviewer [5821 above]. Properties of such algebras are examined in terms of the geometry of the associated projective space, and applications are confined to metrisable algebras of dimensions 3, 4 and 6. A. G. Walker (Liverpool)

5823:

Boclé, Jean. Sur l'existence d'une mesure invariante par un groupe de transformations. C. R. Acad. Sci. Paris 247 (1958), 798-800.

A locally compact group E with left invariant Haar measure λ on its Borel sets acts as a measurability preserving transformation group on a space S equipped with a σ -algebra of measurable sets. The author considers conditions which ensure the existence of a measure in S which is invariant under the group. The crucial condition appears to be the existence in E of at least one family $\{U_i, i \in I\}$ of measurable sets indexed by a directed set I with countable base such that

$$\lim_i \lambda(U_i + U_j) / \lambda U_i = 0, \quad \lim_i \lambda(U_i + tU_i) / \lambda U_i = 0 \quad (t \in E),$$

where "+" means the symmetric difference.

P. A. Smith (New York, N.Y.)

5824:

Nôno, Takayuki. On the singularity of general linear groups. J. Sci. Hiroshima Univ. Ser. A 20 (1956/57), 115-123.

A question concerning the exponential mapping of a Lie algebra, L , into its associated Lie group is discussed. This mapping is a homeomorphism for sufficiently small neighborhoods of the zero of L . In the neighborhood of which points $Y \in L$ is $\exp X$ not a homeomorphism? Such points Y are called "singular". Theorems 1 and 2 characterize singular elements by properties of the eigenvalues of associated matrices and properties of the commutator groups of the element. These theorems are applied to the full linear groups over the complex and real numbers. Several results are obtained concerning the regularity of one-parameter groups joining the identity element to an arbitrary element of $\exp L$.

A. J. Coleman (Toronto, Ont.)

5825:

Nôno, Takayuki. Note on the paper "On the singularity of general linear groups." J. Sci. Hiroshima Univ. Ser. A 21 (1957/58), 163-166.

A gap in the proof of Theorem 1 of the paper reviewed above is filled and the nature of the correspondence effected by the exponential mapping is explored in greater detail with the help of the Jordan canonical reduction of $\exp(\text{ad} X)$.

A. J. Coleman (Toronto, Ont.)

5826:

Nôno, Takayuki. On the branches of logarithmic function of a matrix variable. J. Sci. Hiroshima Univ. Ser. A 21 (1957/58), 1-6.

The inverse of the exponential mapping discussed in the two preceding reviews is the logarithmic function of the title. It is a many-valued function. In analogy with analytic continuation theory, results are obtained on the paths by which various branches of the logarithm are connected. One result will be chosen to quote: there exist "arbitrarily small" one-parameter groups connecting two branches.

A. J. Coleman (Toronto, Ont.)

5827:

Kostant, Bertram. A formula for the multiplicity of a weight. Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 588-589.

A theorem is announced concerning the multiplicity of the weights of an irreducible finite-dimensional representation of a complex semi-simple Lie algebra. The formula obtained permits effective calculation of the multiplicity for any weight of any irreducible representation.

A. M. Gleason (Cambridge, Mass.)

TOPOLOGICAL ALGEBRA

5828:

Hu, S. T. Continuous associative multiplications in locally triangulable spaces. Fund. Math. 46 (1958), 109-115.

A. D. Wallace has posed the problem: What compact connected Hausdorff spaces admit a continuous associative multiplication with a two-sided unit? The object of this paper is to investigate this problem on a class of locally triangulable spaces. A point x in a topological space S is said to be a conic point if there exists a neighborhood U of x in S such that the closure of U is topologically the cone over the frontier of U with x as vertex. A point x_0 of a topological space S is said to be unstable if for every open neighborhood U of x_0 in S , there exists a homotopy $d_t: S \rightarrow S$ ($0 \leq t \leq 1$) which satisfies the following conditions: (i) d_0 is the identity map; (ii) $d_t(U) \subset U$ for every $t \in I$; (iii) $d_t(x) = x$ for each $x \in S - U$ and $t \in I$; and (iv) $d_1(S) \subset S - x_0$. An essential multiplication on a topological space is a continuous associative multiplication with two-sided unit which fails to be a topological group operation. The main result of the paper is the following theorem: If a compact connected Hausdorff space S admits an essential multiplication with a conic point u as a two-sided unit, then u is an unstable point of S . As an immediate corollary the author obtains: If S is a compact connected locally triangulable space without unstable points, then S admits no essential multiplication. As a further corollary to the theorem the following result is proved: Given any essential multiplication in a compact connected subspace S of a locally triangulable space X with no unstable point, the maximal subgroup of the unit in S is contained in the frontier of S in X . Special cases of these results overlap and extend known results of Wallace and Mostert and Shields.

A. Lester (New Orleans, La.)

FUNCTIONS OF REAL VARIABLES

See also 5773, 5910, 6068.

5829:

*Милованова, Л. Н. Функции и их исследование. [Milovanova, L. N. Functions and their investigation.] Izdat. Akad. Pedagog. Nauk RSFSR, Moscow, 1958. 124 pp. 1.35 rubles.

A semi-popular and elementary introduction to the theory of real functions of one variable.

5830:

Sierpiński, W. Sur un théorème de S. Saks concernant les suites infinies de fonctions continues. *Fund. Math.* 46 (1958), 117-121.

The theorem in question is: If a set Q has the property L (i.e., if every nondense perfect subset and so every set of the first category contains at most a denumerable subset of Q), then there exists a uniformly bounded sequence of continuous functions such that every subsequence $f_{m_n}(x)$ is convergent at most at a denumerable subset of Q . This note corrects an error which appears in the proof in the author's *Hypothèse du continu* [Warsaw-Lwow, 1934, p. 47] by showing how to construct a sequence of continuous functions $f_n(x)$ such that $0 \leq f_n(x) \leq 1$ for all n and x , and for every interval I of length $> 1/n$ and $h < n$ there exists a real number x in I such that $|f_n(x) - f_h(x)| \geq \frac{1}{n}$. Such a sequence then has the property that for every subsequence $f_{m_n}(x)$ of $f_n(x)$ the points of convergence form a set of the first category.

T. H. Hildebrandt (Ann Arbor, Mich.)

5831:

Gesztyeli, Ernő. Eine neue Begründung der Differentialrechnung. *Mat. Lapok* 9 (1958), 91-114. (Hungarian. Russian and German summaries)

The considerations of the author are based on the ring of "mixed quantities" in the sense of L. Neder [Math. Ann. 118 (1941), 251-262; MR 5, 257]. With the aid of well-defined "infinitesimal" quantities the author introduces, without using limiting processes, the concepts of differentiability, differential and differential quotient, showing that these coincide with the corresponding classical concepts. The paper seems to be related to more general investigations of E. Kähler [Bericht über die Mathematiker-Tagung in Berlin, 1953, pp. 58-163, Deutsch. Verlag der Wissenschaften, Berlin, 1953; MR 16, 563].

A. Kertész (Debrecen)

5832:

Wilansky, Albert. A definite integral. *Amer. Math. Monthly* 65 (1958), 770-771.

A derivation is given for the existence and value of the integral $\int_a^b x^p dx$ without first assuming existence. Also, an inequality is given which connects the integral and the Riemann sum for an arbitrary partition.

E. Frank (Chicago, Ill.)

5833:

Bogdănescu, V. On the extension of the derivation and the integration by parts. *Lucrarile Inst. Petrol Gaze București* 4 (1958), 253-273. (Romanian. Russian and English summaries)

The author defines

$$\left| \int_a^x F(x) dx = F(x_1) \right. \quad \text{and} \quad \left| \int_{x_1}^x F(x) dx = F(x_2) - F(x_1), \right.$$

with analogous notation for functions of several variables. He expresses formulas for differentiating integrals in a form involving these substitution operators and also obtains formulas for manipulating combinations of definite integrals and substitution operations.

R. P. Boas, Jr. (Evanston, Ill.)

5834:

Levin, V. I. Definitions of elementary transcendental functions by means of integral representations. *Tul'sk. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki* 5 (1954), 76-106. (Russian)

5835:

Zeitlin, David. A Wronskian. *Amer. Math. Monthly* 65 (1958), 345-349.

The following result is proved: If u and v are differentiable functions, then the Wronskian of the set of functions u, uv, uv^2, \dots, uv^n is given by

$$W(u, uv, \dots, uv^n) = u^{n+1} \cdot [Dv]^{n(n+1)} \cdot \prod_{i=1}^n i!$$

This is achieved by showing that the Wronskian determinant can be transformed into triangular form. Several applications are given.

J. Elliott (New York, N.Y.)

5836:

Yarugin, A. N. Investigation of the function

$$I(\theta) = \int_0^\infty p(\theta) L(\theta) d\theta.$$

Vesci Akad. Navuk BSSR. Ser. Fiz.-Tehn. Navuk 1958, no. 3, 57-71. (Byelorussian)

Here $p(\theta)$ is a polynomial in $\sin \theta$ and $\cos \theta$ for which $\int_0^{2\pi} p(\theta) d\theta = 0$, and $L(\theta) \rightarrow 0$ as $\theta \rightarrow \infty$. Investigation is made, for example, of the rapidity of approach to zero of $L(\theta)$ in comparison with that of $I(\theta)$.

5837:

Biernacki, Mieczysław. Sur les polynômes dont tous les zéros sont réels. *Ann. Univ. Mariae Curie-Skłodowska. Sect. A* 10 (1956), 61-75 (1958). (Polish and Russian summaries)

The author establishes the following two theorems.

I. Let a and b be consecutive zeros of a polynomial $f(x)$ possessing only real zeros and let ξ be the point in $a < x < b$ at which $f(x)$ has its extremum. Let $M = \max |f(x)|$, $a < x < b$, and let $0 < h < 1$. Then, if $b - \xi \geq \xi - a$ and $|f(\xi)| = hM$, $\xi < \beta < b$, then $(\beta - \xi)/(b - \xi) \leq (1 - h)^{1/2}$, with the equality holding only for second degree $f(x)$.

II. If, in addition, a is the smallest simple zero of $f(x)$ and $f(a) = f(\beta) = hM$, $a < \alpha < \beta < b$, then $\alpha + \beta \leq a + b$. If $f'(c) = 0$ for $a < c < b$, then $c \leq (a + b)/2$. The equalities hold only for second degree $f(x)$.

The proof of (I) is by choosing $\xi = 0$, introducing the parabola $y = q(x)$ with vertex at $(0, M)$ and through points $(\pm b, 0)$ and studying the function

$$v(x) = [f(x)(x+b)/q(x)(x-a)].$$

These theorems lead to the bounds on the ratio $(\beta - \alpha)/(b - a)$ derived by S. Paszkowski [Ann. Polon. Math. 5 (1958), 165-194] and to an inequality similar to that of Erdős and Grünwald [Ann. of Math. (2) 40 (1939), 537-548] for the area under curve $y = f(x)$ between $x = a$ and $x = b$.

M. Marden (Milwaukee, Wis.)

5838:

Constantinescu, D.; et Constantinescu, F. Sur la séparation des racines de deux polynômes avec toutes les racines réelles. *Gaz. Mat. Fiz. Ser. A (N.S.)* 10(63) (1958), 449-453. (Romanian. French and Russian summaries)

Properties of the separation of the zeros of two polynomials $P(x)$ and $Q(x)$ of the same degree with all zeros real are studied. A proof is given of a theorem of V. A. Markov on the separation of the zeros of $P'(x) = dP(x)/dx$ and $Q'(x) = dQ(x)/dx$, as well as a theorem relative to the minimum length of the intervals between two zeros of $P(x)$ and $Q(x)$. Finally, it is shown that $P'Q - Q'P \neq 0$ is necessary and sufficient for the separation of the zeros of $P(x)$ and $Q(x)$. [Editor's note: Cf. two papers by the

second author: Uspehi Mat. Nauk (N.S.) 12 (1957), no. 6(78), 147-148; C. R. Acad. Sci. Paris 247 (1958), 256-257; MR 20#943, #4621.] E. Frank (Chicago, Ill.)

5839:

Rymarenko, B. A. Forms of multiply-monotonic extremal polynomials. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 35-37. (Russian)

Let $T_n^{(h)}$ and $B_n^{(h)}$ be the classes of polynomials $y_n(x) = \sum_{k=0}^n p_k x^k$, of degree at most n , with real coefficients, which, respectively, are monotonic of order $h+1$ on $[-1, 1]$ or have nonnegative odd derivatives up to the $(2h+1)$ th (inclusive) on the real axis and nonnegative derivatives at $x=-1$ up to the $2h$ th (inclusive); let G_{2m} be the class of functions $\int_{-\infty}^x e^{-z^2} y_{2m}(z) dz$, where $y_{2m}(z)$ is a real polynomial which is nonnegative on the real axis. The author seeks in each of these classes the function with the smallest variation, satisfying one or more conditions of the form $\sum_{k=0}^n \alpha_k p_k = A$. (1) In $T_n^{(h)}$ there is an extremal polynomial satisfying one condition, and it is of the form

$$\int_{-1}^x (x-z)^h (1-z)^\alpha (1+z)^\beta U_m^2(z) dz,$$

with $U_m(z)$ a polynomial of degree at most $\frac{1}{2}(n-1-h-\alpha-\beta)$, all of whose roots are on $[-1, 1]$, and α and β are 0 or 1. (2) In $B_n^{(h)}$ there is an extremal polynomial satisfying two conditions and it is of the form $\int_{-1}^x (x-z)^{2h} U_m^2(z) dz$, where $U_m(z)$ is a real polynomial of degree at most $\frac{1}{2}(n-1-2h)$. (3) In G_{2m} there is an extremal polynomial satisfying two conditions and it is of the form $\int_{-\infty}^x e^{-z^2} U_m^2(z) dz$, where $U_m(z)$ is a real polynomial of degree at most m . Some examples are worked out. R. P. Boas, Jr. (Evanston, Ill.)

5840:

Härtter, Erich. Über die Klasse der Gewichtsfunktionen bei der Verallgemeinerung des Dichtesatzes. J. Reine Angew. Math. 200 (1958), 89-98.

The class K consists of the functions $f(x)$ with the following properties: $f(x)$ is defined, real valued and continuous for $x \geq 0$; $f(x) > 0$ if $x > 0$; $f(x) \leq f(y)$ if $0 \leq x < y$; $f(x) \cdot f(x+2t) \leq (f(x+t))^2$ if $x \geq 0$, $t \geq 0$. The author collects a number of remarks on K . Two examples may suffice. Theorem 6: A polynomial with real non-negative coefficients which has only real roots belongs to K . Theorem 11: Let $f(x)$ be strictly convex; $f(x) \in K$; $f(0) = 0$; $\lim_{t \rightarrow 0} f(2t)/f(t) > 2$. Then $f(x) + c \notin K$ if $c > 0$.

P. Scherk (Saskatoon, Sask.)

MEASURE AND INTEGRATION

See also 5823, 6034.

5841:

Okano, Hatsuo. Measures in the ranked spaces. I, II. Proc. Japan Acad. 34 (1958), 136-141; 205-207.

Let R be a ranked neighborhood space in the sense of K. Kunugi [same Proc. 30 (1954), 553-556, 912-916; MR 17, 389]. Under two sets of assumptions (which are too involved to be here stated) the author defines non-negative functions of certain neighborhoods in R . [The method is similar to Loomis' construction in Duke Math. J. 16 (1949), 193-208; MR 10, 600.] By means of these functions outer measures are defined for which all open

sets are measurable. Some examples are given in the first one of the communications.

H. M. Schaerf (St. Louis, Mo.)

5842:

Ikegami, Teruo. A note on the integration by the method of ranked spaces. Proc. Japan Acad. 34 (1958), 16-21.

In this note K. Kunugi's method of defining integrals of real functions of a real variable [same Proc. 32 (1956), 215-220; MR 18, 567] is generalized to real functions on a locally compact topological group G as follows.

Let V be a compact neighborhood of the unit element of G and ϕ the vector space (over the field of real numbers) generated by the characteristic functions of all open subsets of V whose boundaries are null sets of a left-invariant Haar measure m on G . It is shown that if also the boundary of V is a null set of m , then ϕ can be made into a uniform ranked space so that an extension of the natural elementary integral on ϕ in a way similar to the one in the quoted paper is possible.

H. M. Schaerf (St. Louis, Mo.)

5843:

Small, W. A. A note on defining an extension of a probability measure on subsets of function space, by applying one of J. L. Doob's theorems. Proc. Iowa Acad. Sci. 65 (1958), 332-334.

Remarks on the extensions of probability measures on the space of functions on $(-\infty, \infty)$ relating the extensions of measures of Borel sets (product topology) with the extensions of measures defined on the Borel field of sets generated by the sets of the form $\{w: a < w(t_0) < b\}$. See the general discussion of such measures and extensions by the reviewer [Bull. Amer. Math. Soc. 53 (1947), 15-30; MR 8, 472]. J. L. Doob (Urbana, Ill.)

5844:

Fleming, Wendell H. Functions whose partial derivatives are measures. Bull. Amer. Math. Soc. 64 (1958), 364-366.

The real-valued summable functions $f(x)$, $x = (x_1, \dots, x_N) \in R_N$, of generalized bounded variation in the sense of Tonelli-Cesari have been interpreted in an earlier paper by K. Krickeberg [Ann. Mat. Pura Appl. (4) 44 (1957), 92, 105-133; MR 20#2420] as those whose first partial derivatives in the sense of distribution theory are measures. Another proof of this fact was given by the author in terms of L. C. Young's theory [ibid. 44 (1957), 92, 93-103; MR 20#2421]. Two subclasses of these functions are considered in the present expository paper: the class F_1 of those f for which the same partial derivatives are functions, and the class F_2 of those f whose range is a discrete set. Each function $f \in F_1$, or $f \in F_2$, is equal (up to a set of Hausdorff N -measure zero) to a function f' which is the I -completion of conveniently defined subclasses F_{01} , F_{02} in terms of the theory of N. Aronszajn and K. T. Smith [Ann. Inst. Fourier, Grenoble 6 (1955-1956), 125-185; MR 18, 319], namely, there are sequences f_n of functions of these subclasses which are I -convergent to f' and also pointwise convergent to f' in R_N except in a set of Hausdorff $(N-1)$ -measure zero. L. Cesari (Baltimore, Md.)

5845:

Rivkind, Ya. I. Generating sets of dynamical systems. Grodnen. Gos. Ped. Inst. Uč. Zap. Ser. Mat. 2 (1957), 86-88. (Russian)

Let $f(t, \phi)$ be a dynamical system defined in a given space Ω with an invariant measure μ , with $\mu(\Omega) = 1$, and denote by A_r the set $\sum_{0 \leq t \leq r} f(A, t)$, where A is a gener-

ating set of $f(p, t)$. If for the dynamical system $f(p, t)$ the set A has a positive measure, then almost all the trajectories passing through it are closed. It is also shown as a consequence of this result that for an ergodic dynamical system the set A is non-measurable with respect to the invariant measure.
O. Onicescu (Bucharest)

5846:

*Гохман, Э. Х. Интеграл Стильтьеса и его приложения. [Gohman, E. H. The Stieltjes integral and its applications.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 191 pp. 4.40 rubles.

This book, which is intended to give "an exact, compendious and relatively complete account of the Stieltjes integral", demands from the reader only an acquaintance with the elements of the differential calculus. The Stieltjes integral is defined in the form (somewhat more general than the classical form) due to Šatunovskii, namely: the number J is called the integral from a to b of the function f with respect to the function g if to each positive ε there corresponds a subdivision Q of the interval $[a, b]$ such that for every refinement of Q and every choice of intermediate points we have the inequality

$$|J - \sum (fdg; Q)| < \varepsilon.$$

The Fourier-Stieltjes and Lebesgue-Stieltjes integrals are also discussed and some applications are given to the theory of probability.

5847:

Poenaru, V. Sur la "longueur" d'une courbe continue arbitraire. Acad. R. P. Romine. Stud. Cerc. Mat. 9 (1958), 173-180. (Romanian. Russian and French summaries)

Es seien C ein offenes Intervall, $\varphi(t)$ mit $t \in \bar{C}$ eine stetige Kurve in R^n , A die Menge der Parameterwerte, denen mehrfache Punkte entsprechen und J_1, J_2, \dots die offenen Intervalle, deren Vereinigung $C - A$ ist. Als "berechenbare" Länge der Kurve bezeichnet der Verf. die Summe der Längen der Kurven $\varphi(t)$ mit $t \in J_n$. Dies ist eine unteradditive Funktion von \bar{C} und fällt bei höchstens abzählbarem \bar{A} mit der gewöhnlichen Länge zusammen. Bedeutet hingegen B eine überabzählbare, abgeschlossene Teilmenge von \bar{C} , so existiert eine Kurve mit $A \subset B$, deren berechenbare Länge kleiner als ihre gewöhnliche Länge ist.
K. Krickeberg (Heidelberg)

FUNCTIONS OF A COMPLEX VARIABLE

See also 5894, 5968, 5999, 6081.

5848:

Myškis, A. D. Further remarks about the problem of N. N. Luzin. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 2(74), 155-157. (Russian)

Let $f(z)$ be continuous on a region D . For given $z \in D$ the set of monogenicity \mathfrak{M}_z of f is defined as the intersection of the closures of all sets M_ε with $\varepsilon > 0$ of values of the quotient $[f(z + \Delta z) - f(z)]/\Delta z$ for all Δz with $0 < \Delta z \leq \varepsilon$. In connection with a problem stated by Luzin, it was proved by Yu. Yu. Trohimčuk [same Uspehi 11 (1956), no. 5(71), 215-222; MR 19, 257] that if \mathfrak{M}_z does not depend on $z \in D$, then \mathfrak{M}_z is either a single point, the circumference of a circle, or the entire complex plane, the

question being left open whether the third possibility actually occurs. In the present article a realization of the third case is constructed.
From the introduction

5849:

Jenkins, James A. On the Denjoy conjecture. Canad. J. Math. 10 (1958), 627-631.

The author proves the following generalization of the Denjoy conjecture: If the integral quasiconformal mapping $f(z)$ of maximal dilation K has n distinct asymptotic values and $M(r)$ denotes the maximum modulus of $f(z)$ for $|z|=r$, then $\liminf_{n \rightarrow \infty} M(r)r^{-n/2K} > 0$.

The author's proof is especially noteworthy in that with one minor exception no use is made of the principle of harmonic majorization. Otherwise, the proof is completely carried out using only methods from the theory of extremal length.
H. L. Royden (Zürich)

5850:

Videnskii, V. S. A generalization of V. A. Markoff inequalities. Dokl. Akad. Nauk SSSR 120 (1958), 447-449. (Russian)

Let $a > 0$. For polynomials of degree $\leq n$ which do not exceed $|ax + i\sqrt{1-x^2}|$ for $-1 \leq x \leq 1$, the author determines the greatest possible value of the maximum modulus of the k th derivative in that interval.

L. C. Young (Cambridge, England)

5851:

König, Heinz. Zur Charakterisierung der positiven rationalen Funktionen. Arch. Math. 8 (1957), 409-412.

A holomorphic function is called a positive function if it possesses a positive real part in the half plane $\operatorname{Re}(s) > 0$ and is real-valued for all real $s > 0$. The point set $E \in R^n$ is called algebraically characterisable if there are finitely many polynomials $A_1(x_1, \dots, x_n), \dots, A_r(x_1, \dots, x_n), B_1(x_1, \dots, x_n), \dots, B_s(x_1, \dots, x_n)$ with real coefficients in n unknowns with the property that the point $(c_1, \dots, c_n) \in R^n$ belongs to E if the inequalities $A_1(c_1, \dots, c_n), \dots, A_r(c_1, \dots, c_n) > 0, B_1(c_1, \dots, c_n), \dots, B_s(c_1, \dots, c_n) \geq 0$ are satisfied. The point set $E(m, n)$ consists of all points $(a_0, \dots, a_m, b_0, \dots, b_n) \in R^{m+n+2}$ with $a_m b_n \neq 0$ and with the property that the rational function $(*) Z(s) = (a_0 + a_1 s + \dots + a_m s^m)(b_0 + b_1 s + \dots + b_n s^n)^{-1}$ is a positive function.

The author uses a criterion which he attributes to Cauer [Math. Ann. 106 (1932), 369-394] to establish the following theorem: the point set $E(m, n)$ is not algebraically characterisable in the case $|m-n| \leq 1$ and $\min(m, n) > 1$. The criterion is as follows: The rational function $(*)$ with real coefficients and $a_m b_n \neq 0$ is a positive function if it is holomorphic in $\operatorname{Re}(s) > 0$, has at most poles of the first order with positive residues on the imaginary axis (including the point at infinity), and has at all regular points of the imaginary axis a non-negative real part $\operatorname{Re}(Z(i)) \geq 0$. The set $E(m, n)$ is empty if $|m-n| > 1$ and is characterisable if $|m-n| \leq 1, \min(m, n) \leq 1$.

W. C. Royster (Lexington, Ky.)

5852:

Perron, Oskar. Über zwei ausgeartete Heinesche Reihen und einen Kettenbruch von Ramanujan. Math. Z. 70 (1958), 245-249.

The author considers two series,

$$G(b, q, x) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2) \cdots (1-q^n)} \times \frac{b^n x^n}{(1+b)(1+bq) \cdots (1+bq^{n-1})}$$

$$H(b, q, x) = 1 + \sum_{p=1}^{\infty} \frac{q^p}{(1-q)(1-q^2) \cdots (1-q^p)} \times \frac{(-x)^p}{(1+b)(1+bq) \cdots (1+bq^{p-1})},$$

which are limit cases of the Heine series. (1) Between these series exist the relations $G(b, q, x) = H(b^{-1}, q^{-1}, qx)$, $H(b, q, x) = G(b^{-1}, q^{-1}, qx)$. (2) For $0 < |q| < 1$, $G(b, q, x)$ is an entire function of x . For $|q| > 1$, if $b=0$, it is a constant 1, if $b \neq 0$ it converges only in the circle $|x| < 1$. It is shown that in this case G can be continued analytically as a meromorphic function of x , and G is also expressed as the quotient of two entire functions of x . (3) For $|q| > 1$, $H(b, q, x)$ is an entire function of x . For $0 < |q| < 1$ it converges only in the circle $|x| < |q|^{-1}$. It is shown, however, that H can be analytically continued as a meromorphic function of x , and H is also expressed as the quotient of two entire functions of x . (4) It is shown that

$$(1+b)G(b, q, x)/G(bq, q, x) = (1+b)H(b, q, x)/H(bq, q, x) \\ = 1 + b + \frac{bqx}{1+bq} + \frac{bq^2x}{1+bq^2} + \frac{bq^3x}{1+bq^3} + \cdots,$$

from which can be easily obtained the Ramanujan continued fraction

$$1 + b + \frac{xq}{1+bq} + \frac{xq^2}{1+bq^2} + \frac{xq^3}{1+bq^3} + \cdots = \\ (1+b)G(b, q, xb^{-1})/G(bq, q, xb^{-1}).$$

(5) For certain limit cases infinite product formulas are obtained which show the poles of $G(b, q, x)$ and $H(b, q, x)$.
E. Frank (Chicago, Ill.)

5853:

Mahovikov, V. I. Two mixed boundary problems for analytic functions. *Dopovidi Akad. Nauk Ukraïn. RSR* 1957, 431-435. (Ukrainian. Russian and English summaries)

A method is proposed for solving a boundary problem which consists in determining a function that is analytic within the unit circle and satisfies certain boundary conditions on sets of arcs of the circumference of the unit circle. The boundary conditions are given as systems of equations in terms of given functions and of the sought function and its conjugate. The method consists in reducing each of two given systems of equations to a single equation.
H. P. Thielman (Ames, Iowa)

5854:

Vertgeim, B. A. Approximate construction of some conformal mappings. *Dokl. Akad. Nauk SSSR (N.S.)* 119 (1958), 12-14. (Russian)

The conformal mapping of the circle $|z| < 1$ on a domain in the w -plane which contains $w_0=0$ and has a simple closed boundary L , starlike relatively to w_0 and given by

$$w = \exp\{f(t) + it\}, \quad f(t+2\pi) = f(t),$$

with the condition $w(0)=0$, $w'(0)>0$, leads to a non-linear singular integral equation

$$(*) \quad t(\varphi) - \varphi = S[t(\varphi)]$$

concerning a function $t=t(\varphi)$, $t(\varphi+2\pi)=t(\varphi)+2\pi$. Here S is the integral operator with a Hilbert kernel

$$Sh(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} h(\sigma) \operatorname{ctg} \frac{\varphi-\sigma}{2} d\sigma.$$

The formula (*) leads to iterations

$$t_{n+1}(\varphi) = \varphi + S[t_n(\varphi)],$$

and the process (Theodorsen's method) eventually converges [Cf. S. Warschawski, *Quart. Appl. Math.* 3 (1945), 12-28; MR 6, 207].

The author considers a non-starlike boundary L :

$$w = \exp\{f_1(t) + i f_2(t)\}, \quad f_1(t+2\pi) = f_1(t), \quad f_2(t+2\pi) = f_2(t) + 2\pi, \\ (f_1')^2 + (f_2')^2 \geq d^2 > 0,$$

with f_1 and f_2 twice continuously differentiable and f_1'' and f_2'' satisfying a Lipschitz condition. The mapping problem then leads to a non-linear singular integral equation

$$(**) \quad P[t(\varphi)] = f_2[t(\varphi)] - \varphi - S f_1[t(\varphi)] = 0.$$

Under specific conditions a solution $t^*(\varphi)$ of (**) exists and an iteration formula is given for a sequence $\{t_n(\varphi)\}$ converging to $t^*(\varphi)$.
B. A. Amirà (Jerusalem)

5855:

Fil'čakov, P. F. Numerical method for determining the constants of the Christoffel-Schwarz integral. *Ukrain. Mat. Ž.* 10 (1958), 340-344. (1 insert) (Russian)

In a series of earlier articles [same *Ž.* 7 (1955), 453-470; 8 (1956), 76-91, 299-318; MR 19, 538, 539] the author has developed applications of methods of conformal mapping to problems of filtration. Here a method is given for the numerical determination of the constants in the Christoffel-Schwarz integral for mapping an arbitrary polygon into the upper halfplane. The method depends essentially upon inverting a series, and is illustrated in the case of a quadrilateral. A. S. Householder (Oak Ridge, Tenn.)

5856:

Tamura, Jirô. Prolongation of Riemann surfaces. *Sûgaku* 9 (1957/58), 1-7. (Japanese)

This paper is a survey of the problem of whether there exists a Riemann surface G with a proper subregion which is conformally equivalent to a given Riemann surface F . When this is possible F is called prolongable, otherwise maximal. After a sketch of results obtained so far, the author considers in § 2 a domain with horizontal slits as boundary and a minimum slit domain in a plane. This minimality is equivalent to any of the following propositions: (i) a slit domain Ω belongs to class O_{AD} ; (ii) the span of Ω is 0; (iii) the complement of Ω belongs to N_2 [Ahlfors and Beurling, *Acta Math.* 83 (1950), 101-129; MR 12, 171]. He then constructs a special slit domain due to Koebe. In § 3 he is concerned with an open Riemann surface F with finite genus p , and proves Bochner's theorem which asserts that F admits a prolongation to a closed Riemann surface with genus p . A result by Mori that the prolongation is unique except for conformal equivalence if and only if $F \in O_{AD}$, and a result by Oikawa that the set of all prolongations with genus p of F forms a compact connected set in the Teichmüller space, are mentioned. Next in § 4 a portion F_1 of a Riemann surface F is called an interior of a closed Jordan curve C if C divides F into F_1 and F_2 and if F_1 is of planar character and not simply connected. The set of such C on F will be denoted by $\Gamma(F)$. Mapping F_1 onto a slit domain and sewing up the image at the slits which do not correspond to C , a prolongation of F is obtained. F is called Γ -free if $\Gamma(F)$ is empty; Heins called G a dense prolongation (continuation) of F when F is conformally equivalent to a proper subregion of G which is dense in G . It is proved using the above sewing process that if $\Gamma(F)$ is not empty there exists a Γ -free dense prolongation of F .

Any open Riemann surface is topologically equivalent to some prolongable Riemann surface, as shown by Bochner, but there exists an open Riemann surface which is maximal. The author presents in § 5 his proof [Sci. Papers Coll. Gen. Ed. Univ. Tokyo 7 (1957), 19-22; MR 20 #2435] of the main theorem by Bochner and Heins that any prolongable Riemann surface has a maximal dense prolongation. From his proof it follows that any Γ -free Riemann surface with null boundary is maximal, as remarked by Oikawa. This assures the existence of an open maximal Riemann surface. The examples by Radó and Tsuji are special cases.

Finally, in § 6 conditions for a Riemann surface to be prolongable are discussed. Suppose that the universal covering surface \hat{F} of F is mapped on $|z| < 1$ and every point of $|z| = 1$ is a limit point with respect to the Fuchsian group corresponding to the covering transformation of \hat{F} . The author has shown [ibid. 6 (1956), 123-127; MR 18, 727] that this condition is not sufficient for a Γ -free F to be maximal, contrary to the assertion of de Possel and Heins. Now suppose that a simply connected subregion D of F is mapped onto the right half plane and denote by E the set on the imaginary axis which corresponds to the ideal boundary of F . De Possel called D of maximal type if $E \in N_2$ in the plane, and proved that a Γ -free F is maximal if and only if every simply connected subregion of F is of maximal type. The author wonders if this criterion is useful for deciding whether a given Riemann surface is actually maximal or not. The paper is concluded with the relation with the classification problem. In the following, let F always be Γ -free. Oikawa showed that $F \in O_{HD}$ is maximal and the author showed that F is maximal if there exists a Schottky covering surface $\hat{F} \in O_{AD}$ of F . Sario [Ann. Acad. Sci. Fenn. Ser. A. I, no. 50 (1948); MR 10, 365] raised the following questions: (1) Is every $F \in O_{AD}$ maximal? (2) Does every maximal F belong to O_{AD} ? According to the author, (1) is answered negatively, but (2) remains open.

M. Ohtsuka (Lawrence, Kans.)

5857:

Mori, Shin'ichi. A remark on a subdomain of a Riemann surface of the class O_{HD} . Proc. Japan Acad. 34 (1958), 251-254.

The author proves the following theorem: Let F be a Riemann surface of class O_{HD} which is not of class O_G , and let G be a non-compact sub-domain of F . Then every AD function on G whose real part vanishes on the relative boundary of G is a constant.

(While the author's method of proof is not without interest, it should be pointed out that, in the case where the complement of G is compact, the above theorem is a special case of a theorem of Kuramochi [Osaka Math. J. 7 (1955), 109-127; MR 17, 26]: Let F and G be as above, and suppose that the complement of G is compact. Then every AD function on G is constant.)

H. L. Royden (Zürich)

5858:

Firsakova, O. S. Some problems concerned with interpolation by means of integral functions. Dokl. Akad. Nauk SSSR 120 (1958), 477-480. (Russian)

Let λ_n be complex numbers with $|\lambda_n| \uparrow \infty$. The question at hand is that of the existence of an entire function of finite order ρ with given proximate order $\rho(r)$ such that $f(\lambda_n) = a_n$ whenever

$$(*) \quad \limsup \{ \log |a_n| / |\lambda_n|^{\rho(|\lambda_n|)} \} \leq \rho.$$

The author solves the problem, first for general distri-

bution of the λ_n , then for regular distribution, and finally for the λ_n lying in an angle; in the two latter cases, (*) has to be appropriately modified.

R. P. Boas, Jr. (Evanston, Ill.)

5859:

Džrbašyan, M. M.; and Avetisyan, A. E. Integral representation of some classes of functions analytic in an angular region. Dokl. Akad. Nauk SSSR 120 (1958), 457-460. (Russian)

The authors state further integral representations similar to those of earlier papers by the first author [Mat. Sb. N.S. 33(75) (1953), 485-530; Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 133-190; MR 15, 517; 16, 1102].

R. P. Boas, Jr. (Evanston, Ill.)

5860:

Melman, N. The monotonic argument principle and differentiation of inequalities. Dokl. Akad. Nauk SSSR 120 (1958), 1191-1193. (Russian)

This paper is devoted to the construction of majorants ω of a specific kind for a given function f , with a view to obtaining inequalities of the form $|f'(x)| \leq |\omega'(x)|$ from hypotheses of the form $|f(x)| \leq |\omega(x)|$ [see the author's papers, same Dokl. 71 (1950), 609-612; Uspehi Mat. Nauk. (N.S.) 7 (1952), no. 3(49), 3-62; MR 11, 509; 14, 259]. He now turns to multiple-valued functions; the paper is related to a recent one by Ahiezer and Levin [same Dokl. 117 (1957), 735-738; MR 20 #971]. We deal with a region G consisting of the finite plane with a system β of closed intervals deleted from the real axis; regular boundary points are points which are not endpoints of intervals in β . We consider the universal covering surface \tilde{G} of G and its boundary $\tilde{\beta}$ over β . A single-valued function f , analytic in G (or \tilde{G}) is called admissible if there is a single-valued analytic function \tilde{f} (in the same region) taking values on $\tilde{\beta}$ (or β) conjugate to those of f . An admissible function has the form $f = g + ih$, where g and h are real on β (or $\tilde{\beta}$). The class HB in G or \tilde{G} consists of admissible functions $\tilde{\omega}(z) = u(z) - iv(z)$ having no zeros in common with $\omega(z) = u(z) + iv(z)$ and satisfying $|\tilde{\omega}(z)/\omega(z)| < 1$ in G or \tilde{G} . An admissible f subordinate to ω with $\tilde{\omega}$ in HB has the property that for every t of modulus less than 1 the function $\tilde{\omega}(z) - t\tilde{f}(z)$ belongs to HB in \tilde{G} and its argument decreases along $\tilde{\beta}$; it follows that, for instance, $|\gamma f'(x) + \delta f(x)| \leq |\gamma \omega'(x) + \delta \omega(x)|$ on β when $\Im(\gamma/\delta) \leq 0$. The relative deviation of f from ω in \tilde{G} is the upper bound of $|f(z)/\omega(z)|$ and $|\tilde{f}(z)/\tilde{\omega}(z)|$, when $\tilde{\omega} \in HB$ and f is admissible. Theorem: the smallest relative deviation of f , normed by the condition $\gamma f'(\xi) + \delta f(\xi) = 1$ ($\gamma \neq 0$, ξ a regular point of β) is equal to $|\gamma \omega'(\xi) + \delta \omega(\xi)|^{-1}$ and is attained for (and only for) a function of the form

$$f(z) = c(\gamma \omega'(\xi) + \delta \omega(\xi))^{-1} \omega(z) + (1-c)(\gamma \tilde{\omega}'(\xi) + \delta \tilde{\omega}(\xi))^{-1} \tilde{\omega}(z),$$

$0 \leq c \leq 1$. Let $\{a_n\}$ and $\{b_n\}$ be sequences having ∞ as their only limit point and let all the endpoints of β belong to $\{b_n\}$. Let

$$u(z) = e^{g(z)} \prod (1 - z/a_n) e^{P_n(z/a_n)}, \\ v(z) = -i e^{h(z)} \prod (1 - z/b_n) e^{Q_n(z/b_n)},$$

where g and h are real entire functions, P and Q are Weierstrass products, and $\delta_n = \frac{1}{2}$ if b_n is an endpoint and otherwise $\delta_n = 1$. The author gives a necessary and sufficient condition for $\lambda u(z) - i \mu v(z)$ to belong to HB , and illustrates it with examples.

R. P. Boas, Jr. (Evanston, Ill.)

5861:

Dufresnoy, Jacques. Le problème des coefficients pour certaines fonctions méromorphes dans le cercle unité. Ann. Acad. Sci. Fenn. Ser. A. I, no. 250/9 (1958), 7 pp.

I. Schur [J. Reine Angew. Math. 147 (1917), 205-232; 148 (1918), 122-145] gave necessary and sufficient conditions which the coefficients $u_0, u_1, \dots, u_n, \dots$ of the complex function

$$f(z) = u_0 + u_1 z + \dots + u_n z^n + \dots$$

must satisfy in order that $f(z)$ shall be holomorphic in $|z| < 1$ and $|f(z)| < 1$ there.

The author (jointly with C. Pisot) made similar investigations concerning meromorphic functions. A brief account of their results is given in the present paper. An application was made to the study of the set S of algebraic integers > 1 , all conjugates of which have a modulus < 1 .

B. A. Amirà (Jerusalem)

5862:

Regan, Francis; and Rust, Charles. On natural boundaries of a generalized Lambert series. Math. Mag. 31 (1957/58), 45-50.

Let $F(t) = t/(1-t)$; it is known that if the sequence $\{c_n\}$ is well behaved, one should expect the Lambert series $\sum c_n F(z^n)$ to have the unit circle for a natural boundary. The authors discuss this for the generalized Lambert series $\sum b_n F(a_n z^n)$ introduced by Feld [Ann. of Math. 33 (1932), 139-143]. Their approach is to impose hypotheses which ensure that, for appropriate c_n , $\sum b_n F(a_n z^n) = \sum c_n F(z^n)$, and then to quote the relevant theorem for ordinary Lambert series. (In this connection see also C. H. W. Sedgewick [Tohoku Math. J. 43 (1937), 314-325], who studied the series $\sum c_n F(z^n)$ for meromorphic F .)

R. C. Buck (Stanford, Calif.)

5863:

Nabeshima, Ichirô. On angular derivatives. Sûgaku 8 (1956/57), 149-151. (Japanese)

The author continues his work on angular derivatives [for the previous work see Sûgaku 2 (1949), 217-220; 4 (1952), 228-230]. In this review let $f(z)$ be an analytic function in $|z| < 1$ such that $|f(z)| < 1$ and $\lim_{r \rightarrow 1} f(r) = i$ as $r \rightarrow 1$, except for the last part. Denote by D_f the angular derivative at $z=1$ of $f(z)$. The author first proves $D_g \leq D_f$ when $f(z)$ is subordinate to $g(z)$ in a certain manner. Applying this result it is shown that $D_f \leq r(2r-1)^{-1}$ for $w=f(z)$ if $f(0)=0$ and if the image of $|z| < 1$ includes the disc $|w-r| < r$ ($> \frac{1}{2}$). Next he is concerned with relations of angular derivatives with Bergman's kernel functions. Suppose that $w=f(z)$ is univalent, and denote by G the image (included in $|w| < 1$). Set $f(0)=\zeta$ and $\varphi=\arg f'(0)$, and let $K_G(w, \zeta)$ be the Bergman's kernel function for G . The author proves that if $D_f < \infty$, then some end part of every Stolz's path at $w=1$ is included in G ; that $0 < \lim_{w \rightarrow 1} \{\exp(-i\varphi)\} K_G(w, \zeta) < \infty$ as $w \rightarrow 1$ along any Stolz's path; and that, conversely, if this limit exists as $w \rightarrow 1$ arbitrarily and is different from 0 and ∞ , then $D_f < \infty$. For the same $f(z)$ (but with $f(0)=\zeta \neq 0$ this time) the relation $\lim_{w \rightarrow 1} |K_G(w, \zeta)| \leq (1+|\zeta|)^2 K_G(\zeta, \zeta)$ is given, where $w \in G \rightarrow 1$ along any Stolz's path. Finally, let $f(z)$ be a (multivalent) mapping function onto $|w| < 1$ of a multiply connected domain bounded by $|z|=1$ and a finite number of analytic curves and points in $|z| < 1$, and denote by $\varphi(w)$ the inverse function (according to the agreement made at the beginning of this review, $\varphi(w)$ should be written as $f(z)$). The author proves that $D_\varphi = \infty$ at fixed points on $|w|=1$ defined with respect to the automorphic

function $\varphi(w)$ and that $D_\varphi < \infty$ at non-singular points on $|w|=1$ (angular derivatives are considered not only at $w=1$). He then applies these results to prove some properties of the derivative of $f(z)$.

M. Ohtsuka (Lawrence, Kans.)

5864:

Hsu, Tsen-Fang. On the third coefficient of bounded schlicht functions. Advancement in Math. 2 (1956), 279-289. (Chinese)

5865:

Garabedian, P. R.; and Schiffer, M. A proof of the Bieberbach conjecture for the fourth coefficient. Advancement in Math. 3 (1957), 167-200. (Chinese)

A translation from the English in J. Rational Mech. Anal. 4 (1955), 427-465 [MR 17, 24].

5866:

Shich, Shih-yueh. On the coefficients of schlicht functions. Advancement in Math. 3 (1957), 597-601. (Chinese)

5867:

Dundučenko, L. O. On univalent functions parabolically convex in a circle. Dopovidi Akad. Nauk Ukrain. RSR 1958, 128-130. (Ukrainian. Russian and English summaries)

The author studied unifoliate parabolically convex functions in a unit circle, the circle being transformed in a π -domain. A π -domain is defined as one possessing the following properties: 1) it lies in a plane notched along the non-negative part of the real axis, 2) if the point $a = \bar{a} + i\beta$ belongs to the π -domain, then the point $\bar{a} = a - i\beta$ also belongs to it together with the section of the parabola $y^2 = 4c^2x + 4c^4$ ($0 \leq c < +\infty$) joining the two points.

A structural formula is derived for the class of functions studied $w = -1 - 2z + a_2 z^2 + \dots$ expressing the necessary and sufficient condition of the given function of the given class belonging to the given domain.

The author also gives precise appraisals of the modulus of the function and its derivative and the moduli of the Taylor development coefficients. Analogous appraisals are given for the more general class of parabolic convex functions which are normed only by the condition $f(0) = -1$.

Author's summary

5868:

Dundučenko, L. O. On a class of functions univalent in the circle $|z| < (\sqrt{2})^{-1}$. Dopovidi Akad. Nauk Ukrain. RSR 1958, 595-597. (Ukrainian. Russian and English summaries)

A study is made of a particular class of functions, w , holomorphic in the unit circle and univalent in the circle $|z| < 1/\sqrt{2}$. Precise upper and lower estimates are given for $|w|$, $|w'|$, $|\arg(w/z)|$, $|\arg(w'/z)|$ and $|a_n|$, where a_n is the coefficient of z^n in the Maclaurin series of the function.

H. P. Thielman (Ames, Iowa)

5869:

Suvorov, G. D. On distortion of distances in univalent mappings of closed regions. Mat. Sb. N.S. 45(87) (1958), 159-180. (Russian)

Let D be a finite, simply-connected region of the z plane, containing the origin. Using the spherical metric (R) induced by a Riemann sphere of radius R , tangent to the z -plane at the origin, the author defines a metric $\rho(z_1, z_2; D; R)$ for points $z_1, z_2 \in D$, $z_k \neq 0$, as follows. Put $\rho_1(z_1, z_2) = (R) \cdot \text{length of shortest line in } D \text{ connecting } z_1 \text{ to } z_2$;

put $\rho_2(z_1, z_2) = (R)$ -length of shortest crosscut dividing D into two regions, one of which contains 0, the other z_1 and z_2 ; let

$$\bar{\rho}(z_1, z_2) = \min(\rho_1, \rho_2),$$

$$\rho(z_1, z_2; D; R) = \inf\{\bar{\rho}(z_1, z) + \bar{\rho}(z, z_2) | z \in D, z \neq 0\}.$$

If z_k is 0, or a boundary point, ρ is defined as a limiting value. The basic theorem of the paper, which is too complicated to state fully here, considers a homeomorphism $w = T(z)$, $T(0) = 0$, of class C_k , mapping D onto a region Δ , and relates $\rho[T(z_1), T(z_2); \Delta; r]$ to $\rho(z_1, z_2; D; R)$. (For a definition of the class C_k see Dokl. Akad. Nauk SSSR 108 (1956), 777-779 [MR 18, 724].) One of the corollaries is a generalization of a result of Lelong-Ferrand [Représentations conformes et transformations à intégrale de Dirichlet bornée, Gauthier-Villars, Paris, 1955; MR 16, 1096]. E. Reich (Minneapolis, Minn.)

5870:

Hornich, Hans. Zur Frage der isolierten schlichten Funktionen. Math. Ann. 135 (1958), 189-191.

As a contribution to the question of the existence of isolated univalent functions in the space of all functions regular in the unit circle [see the author's previous paper, Abh. Math. Sem. Univ. Hamburg 22 (1958), 38-49; MR 20 #980], the author gives an example of a function $f(z)$, univalent in $|z| < 1$, which has the property that the functions $f(z) + cz$ are non-univalent for all sufficiently small (complex) values of c , except possibly when c is positive. The example is obtained essentially by forming the integral of a function $w = w(z)$ which maps the circle $|z| < 1$ conformally on the "slit" circle $|w| < 1$, $-\pi < \arg w < \pi$.

F. Herzog (East Lansing, Mich.)

5871:

Kreyszig, Erwin; and Todd, John. The radius of univalence of the error function. Bull. Amer. Math. Soc. 64 (1958), 363-364.

The authors determine the largest value of ρ for which $\int_0^\rho e^{-t^2} dt$ is univalent for $|z| < \rho$.

Z. Nehari (Pittsburgh, Pa.)

5872:

Kuroda, Inao. On properties of Friedman's functions and the other functions closely connected with them. Bull. Yamagata Univ. (Nat. Sci.) 4 (1957), 1-11. (Japanese. English summary)

Friedman [Duke Math. J. 13 (1946), 171-177; MR 8, 22] announced the following notable theorem: Let

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n + \dots$$

be a function regular and schlicht in the unit circle $|z| < 1$, and let a_2, a_3, \dots , the coefficients in the power series expansion of $f(z)$, be positive or negative integers or zero; then $f(z)$ is one of the following nine functions:

$$z, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2}, \frac{z}{(1 \pm z)^2},$$

Recently, Viktors Linis [Amer. Math. Monthly 62 (1955), 109-110; MR 16, 809] has generalized the theorem, assuming that the coefficients a_n are the Gaussian integers.

Friedman's functions have such profound meanings and are so important that they are frequently cited throughout the theory of univalent functions. In this paper the writer endeavours to investigate a little further their generation and their remarkable properties, together with many other functions such as $\tan^{-1} z$, $\tanh^{-1} z$, ... closely connected with them; in particular, he tries to explain

several noteworthy relations among them, from the point of view of conformal representation.

From the author's summary

5873a:

Goodman, A. W. Variation of the branch points for an analytic function. Trans. Amer. Math. Soc. 89 (1958), 277-284.

5873b:

Goodman, A. W. On variation formulas for univalent functions. Trans. Amer. Math. Soc. 89 (1958), 285-294.

Let $w = f(z)$ ($f(0) = 0$, $f'(0) > 0$) be a regular analytic function in $|z| < 1$ which maps $|z| < 1$ onto a Riemann domain R , where R is assumed to have at least one simple branch point, say at $w = B$. The author considers a small variation of R which moves this branch point to $B + \lambda$ and leaves the boundary of R and the possible other branch points fixed. If $w = f^*(z)$ ($f^*(0) = 0$, $f^{*'}(0) > 0$) is the function mapping $|z| < 1$ onto this varied domain R^* , the author computes $f^*(z)$ up to a correction term of order $|\lambda|^2$. The basic formalism of the computation is similar to that used by Schiffer in his work on p -valent functions [Amer. J. Math. 65 (1943), 341-360; MR 4, 215]. A corresponding formula is also obtained for the case in which the branch point in question is of second order. The case of higher-order branch points seems to defy computation.

In the second paper, this variational formula is applied to a number of extremal problems, both for univalent and multivalent functions. To make the formula applicable to univalent functions, the introduction of an artificial branch point becomes necessary. This is accomplished by considering instead of $f(z)$ the function $f(z) + A f^2(z)$, where A is a suitably chosen constant. In this manner, a new derivation of the Schiffer differential equation for the normalized univalent function with the largest n th coefficient is obtained. Other applications of the basic formula include a new proof of Schiffer's variational formula for p -valent functions [see the above reference].

Z. Nehari (Pittsburgh, Pa.)

5874:

Goodman, A. W. On the critical points of a multivalent function. Trans. Amer. Math. Soc. 89 (1958), 295-309.

Let $f(z)$ be regular and p -valent in $|z| < 1$. Using a compactness argument, the author shows that, for each $k \geq p$, there exists a largest number $R(p, k)$ in $(0, 1)$ such that the closed disk $|z| \leq R(p, k)$ cannot contain more than k critical points of $f(z)$ if these points are counted according to their multiplicities. With the help of variational methods, certain properties of the extremal functions related to this theorem are derived. The explicit form of the extremal functions is conjectured and shown to possess these properties. A similar discussion is given for the case in which $f(z)$ is of the form $zg(z)$, where $g(z)$ is regular and does not vanish in $|z| < 1$.

Z. Nehari (Pittsburgh, Pa.)

5875:

Hayman, W. K.; and Kennedy, P. B. On the growth of multivalent functions. J. London Math. Soc. 33 (1958), 333-341.

Let $f(z)$ be mean p -valent in $|z| < 1$ and suppose that in addition

$$\limsup_{r \rightarrow 1} (1-r)^{2p} M(r, f) > 0,$$

where $M(r, f) = \max_{|z|=r} |f(z)|$. It is known that under

these conditions there is a θ_0 such that

$$\liminf_{r \rightarrow 1} (1-r)^{2p} / |(re^{i\theta})| > 0$$

and that if $\eta > 0$ then for $\eta \leq |\theta - \theta_0| \leq \pi$

$$\log |(re^{i\theta})| < O \left\{ \left(\log \frac{1}{1-r} \right)^4 \right\}$$

uniformly as $r \rightarrow 1$. The authors prove that O can be replaced by o and that the new estimate is then best possible.

A. W. Goodman (Lexington, Ky.)

5876:

Myrberg, P. J. Iteration der reellen Polynome zweiten Grades. Ann. Acad. Sci. Fenn. Ser. A. I, no. 256 (1958), 10 pp.

Für die Iteration des Polynoms $y^2 + p$ wird für gewisse p der Rand des Attraktionsgebietes von ∞ untersucht: er besteht für $p \geq 1$ aus diskreten Punkten, die im Kreisring $1/2 < |y| < \sqrt{p+1}$ liegen, und für $-3/4 < p < 1/4$ sowie für $-5/4 < p < -3/4$ aus einer Menge geschlossener Kurven, was vermutlich auch für $-2 < p < -5/4$ gilt.

H. Tietz (Münster)

5877:

Kočaryan, G. S. On a generalization of the series of Laurent and Fourier. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk 11 (1958), no. 1, 3-14. (Russian. Armenian summary)

The author uses Faber polynomials to generalize Laurent series to doubly-connected regions bounded by two closed curves and to generalize Fourier series to an expansion of a function defined on a single closed curve. The Faber polynomials are obtained from a mapping of the outside of the inner curve on the exterior of a circle, and a second set is obtained in a similar way from a mapping of the inside of the outer curve on the exterior of a circle. The author shows that a function that is analytic in a doubly-connected region between two level curves, one for each mapping, is representable by the analogue of a Laurent series. Under a supplementary condition on the curves [Al'per, Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 423-444; MR 17, 729] the author shows that a function that is analytic in the doubly-connected region and satisfies a Lipschitz condition of order α in the closed region is approximable to order $n^{-\alpha}$ by generalized Laurent polynomials, and to order $n^{-\alpha} \log n$ by the partial sums of its generalized Laurent series. The same is true for a function that belongs to Lip α on a single simple closed curve. (The author was evidently unaware of the work of H. Tietz on generalized Laurent series: Math. Ann. 129 (1955), 431-450; Michigan Math. J. 4 (1957), 175-179 [MR 17, 251; 19, 1045].)

R. P. Boas, Jr. (Evanston, Ill.)

5878:

Lee, K. P. A fundamental theorem on interpolation of integral functions. Acta Math. Sinica 7 (1957), 268-270. (Chinese)

Applying the Boutroux-Cartan theorem, the author proves the following theorem. Let $f(z)$ be an integral function and $M(r, f)$ its maximum modulus on $|z|=r$. Let $\{a_n\}$ be a sequence of points on z -plane which satisfy the following conditions:

$$(1) \quad |a_n| = \xi_n \quad (n=0, 1, 2, \dots)$$

such that

$$\xi_{n+1} - \xi_n > 2\epsilon h_n \quad (>0), \quad h_n \rightarrow +\infty;$$

$$(2) \quad \lim_{n \rightarrow \infty} (\xi_n/h_n)^{1/n} = 0.$$

Let

$$(I) \quad \limsup_{n \rightarrow \infty} M(r_n \xi_n, f)^{1/n} \frac{\xi_n}{h_n} < 1, \quad 1 < r_n < \frac{\xi_{n+1}}{\xi_n}.$$

Denote

$$f_n(z) = \sum_{m=0}^n f(a_m) \frac{\Pi_n(z)}{(z-a_m) \Pi'_n(a_m)},$$

where $\Pi_n(z) = \prod_{m=0}^n (z-a_m)$. Then $\{f_n(z)\}$ converges uniformly to $f(z)$ as limit in the whole finite plane. Condition (I) may be replaced by

$$\limsup_{n \rightarrow \infty} M(\xi_{n+1}, f)^{1/n} \frac{\xi_n}{h_n} < 1.$$

T. K. Pan (Norman, Okla.)

5879:

Mikeladze, Š. E. Approximate solution of a system of non-linear equations. Soobšč. Akad. Nauk Gruzin. SSR 20 (1958), 647-653. (Russian)

If $F(z)$, $f(z)$ are regular functions of the complex variable z in a domain G and $f'(z_0) \neq 0$, $z_0 \in G$, then

$$F(z) = \sum (m!)^{-1} y^{(m)}(z_0) [f(z) - f(z_0)]^m$$

in a neighborhood of z_0 , where $y^{(0)}(z_0) = F(z_0)$, $y^{(1)}(z_0) = [f'(z_0)]^{-1} (d/dz_0) F(z_0)$, etc. If $F(z) = z$, $f(z) = 0$, α sufficiently close to z_0 , then

$$\alpha = \sum (-1)^m (m!)^{-1} y^{(m)}(z_0) [f(z_0)]^m.$$

[For previous derivations of this series see E. Schröder, Math. Ann. 2 (1870), 317-365; Š. E. Mikeladze, Izv. Akad. Nauk SSSR 4 (1935), 559-586; D. R. Blaskett and H. Schwerdtfeger, Quart. Appl. Math. 3 (1945), 266-268; MR 7, 218; F. Casale, Inst. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83) (1950), 727-734; MR 13, 782, 1140.]

If $F_k(w_1, \dots, w_m)$, $k=1, \dots, m$, are regular functions of w_1, \dots, w_m , and there is a point $(\bar{w}_1, \dots, \bar{w}_m)$ where $F_k=0$, $k=1, \dots, m$, let (w_{10}, \dots, w_{m0}) be a point sufficiently close to it. Let us introduce the auxiliary variable t , two arbitrary numbers t_0, t_1 , and constants $\lambda_1, \dots, \lambda_m$ by means of the relations

$$(*) \quad F_k(w_1, \dots, w_m) - F_k(w_{10}, \dots, w_{m0}) = \lambda_k(t - t_0),$$

$$-F_k(w_{10}, \dots, w_{m0}) = \lambda_k(t_1 - t_0) \quad (k=1, \dots, m).$$

If a certain Jacobian is not null and if we think of $w_k(t)$ as functions of t such that $w_k(t_1) = \bar{w}_k$, $w_k(t_0) = w_{k0}$, $k=1, \dots, m$, then the derivatives $w_k^{(s)}(t_0)$ ($k=1, \dots, m$; $s=0, 1, 2, \dots$) can be obtained from (*) by differentiating and solving linear systems. Then

$$\bar{w}_k = w_k(t_1) = w_k(t_0) + (t_1 - t_0) w_k'(t_0) + \dots \quad (k=1, \dots, m).$$

Numerical examples are given.

L. Cesari (Baltimore, Md.)

5880:

Bishop, Errett. The structure of certain measures. Duke Math. J. 25 (1958), 283-289.

Let X be a compact plane set, ν a complex-valued Borel measure on X . For each continuous function h defined on X , say that " ν is orthogonal to h " if $\int h d\nu = 0$. In studying uniform approximation on X by functions in some given family, it is a key problem to describe those measures which are orthogonal to all functions in the family. F. and M. Riesz proved [in the Proceedings of the 4th Scandinavian Mathematical Congress, 1916] that if X is the circle $|z|=1$ and ν a measure on X orthogonal to all polynomials, then there exists an analytic function H in $|z| < 1$ with radial boundary values $H(e^{i\theta})$ existing

a.e. and summable on $(0, 2\pi)$, with $v = H(e^{it})dz$. Also H is in Hardy's class H^1 . The author generalizes this result as follows: Let C be a compact set having a connected complement, and with boundary X and connected interior U . Let ν be a measure on X orthogonal to all polynomials. Then there exists an analytic function g on U which represents ν in the sense that there is a sequence $\{y_n\}$ of simple closed rectifiable curves in U , converging to the boundary X , with $\int h d\nu = \lim_{n \rightarrow \infty} \int_{y_n} h(z) g(z) dz$ for every h continuous on C . The proof makes use of conformal mapping of $|z| < 1$ onto U . After some related further results about the measure ν , the author obtains the following applications. Th: Every real-valued continuous function on X can be uniformly approximated on X by real parts of polynomials. It follows that the Dirichlet problem has a solution in this case. Th: The algebra of functions on X which are uniform limits of polynomials is a maximal subalgebra of the algebra of all continuous functions on X . This generalizes a result of the reviewer for the case of the circle [Proc. Amer. Math. Soc. 4 (1953), 866-869; MR 15, 440]. J. Wermer (Providence, R.I.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 5963, 5979, 6081.

5881:

Springer, George. Interpolation problems for functions of several complex variables. J. Math. Mech. 7 (1958), 957-962.

Let L^2 denote the class of holomorphic analytic functions f in a $(2n)$ -dimensional domain D for which $\|f\|^2 = \int_D |f|^2 dV < \infty$, where dV is the $2n$ -dimensional volume element. As shown by Bergman [The kernel function and conformal mapping, Amer. Math. Soc., New York, 1950; MR 12, 402], the problem of minimizing $\|f\|$ for $f \in L^2$ and f having a specified value at a point of D is solved by the Bergman kernel function.

As the author points out, an analytic function of n complex variables assumes a given value, not at isolated points, but on an analytic manifold of complex dimension $n-1$, and it is therefore natural to consider interpolation problems in which the functions of L^2 are normalized to assume prescribed values on some given $n-1$ dimensional manifolds. This program is carried out in detail for different types of domains. The extremal functions are completely characterized, and related to suitable kernel functions. Z. Nehari (Pittsburgh, Pa.)

5882:

Ronkin, L. I. Integral functions of finite degree and functions of completely regular growth (of several variables). Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 211-214. (Russian)

The author uses some of his previous results [same Dokl. 92 (1953), 887-890; Mat. Sb. N.S. 39(81) (1956), 253-266; MR 15, 613; 18, 569], extended to n variables, to establish further results, the principal one being a generalization of Titchmarsh's theorem on the indicator diagram of a product, said to differ in form and proof but to be equivalent in content to that given by Mikusiński and Ryll-Nardzewski [Studia Math. 13 (1953), 62-68; MR 15, 408]. The author does not refer to Lions [C. R. Acad. Sci. Paris 232 (1951), 1530-1532; MR 13, 231],

where complex-variable methods were used for the same purpose. The result states (in condensed form) that if K_f denotes the supporting function of the smallest bounded convex set outside which a given f vanishes, we have $K_f + K_g = K_{fg}$. R. P. Boas, Jr. (Evanston, Ill.)

5883:

Alizenberg, L. A. Temlyakov integrals and the boundary properties of analytic functions of two complex variables. Dokl. Akad. Nauk SSSR 120 (1958), 935-938. (Russian)

In the (w, z) -space we consider the domain D with the boundary $|w| = r_1(\tau)$, $|z| = r_2(\tau)$, where $r_1(0) = 0$, $0 < r_1'(\tau) \leq \tau^{-1} r_1(\tau)$, $r_1(1) < \infty$,

$$r_2(\tau) = \exp\left(-\int_0^\tau \tau(1-\tau)^{-1} d \ln r_1(\tau)\right).$$

We consider a function $f(w, z)$, analytic in D , and the function $F = f + w/w' + z/z'$. With the notation

$$u = (r_1(\tau))^{-1} \tau w + (r_2(\tau))^{-1} (1-\tau) z e^{it}$$

and under obvious continuity assumptions on the boundary we have Temlyakov's representations [Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957), 89-92; MR 19, 25]

$$f(w, z) =$$

$$(4\pi^2)^{-1} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|z|=1} (\zeta - u)^{-1} F(r_1(\tau)\zeta, r_2(\tau)\zeta e^{-it}) d\zeta,$$

$$f(w, z) =$$

$$(4\pi^2)^{-1} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|z|=1} (\zeta - u)^{-2} \zeta f(r_1(\tau)\zeta, r_2(\tau)\zeta e^{-it}) d\zeta.$$

The author states without proof several theorems concerning these representations. If the function F in the first integral is replaced by any function $F(\tau, t, \zeta)$ continuous on the boundary of D , the integral yields a function analytic in D . In the special case where D has the form $a|w| + b|z| < 1$, the integral will also yield an analytic function in the domains where $a|w| - b|z| > 1$, but a nonanalytic continuous function where $a|w| + b|z| > 1$, $a|w| - b|z| < 1$. Both of Temlyakov's representations are converted into integrals of Poisson type. It is further stated that the second integral representation holds if and only if

$$\sum_{j=0}^{\infty} \left| \sum_{m=0}^j a_{m,j-m} (r_1(\tau))^m (r_2(\tau))^{j-m} e^{imt} \right|^2$$

converges for almost every (t, τ) .

H. Tornehave (Copenhagen)

5884:

Temlyakov, A. A. Representations of functions of two complex variables by integrals. Dokl. Akad. Nauk SSSR 120 (1958), 976-979. (Russian)

In the (w, z) -space we consider the class of domains D bounded by hypersurfaces $|w| = r_1(\tau)$, $|z| = r_2(\tau)$; $r_1(0) = 0$, $r_1'(\tau) > 0$, $r_1(1) < \infty$, $0 < \tau \leq 1$,

$$r_2(\tau) = \exp\left(-\int_0^\tau \tau(1-\tau)^{-1} d \ln r_1(\tau)\right).$$

The author proves that this is exactly the class of Reinhardt circular regularity domains bounded by twice differentiable surfaces. Let $F(w, z)$ be analytic in D and F, F_w', F_z' continuous in D . With the notations

$$w = \left[\sup_{0 < \tau < 1} \frac{d \ln r_1(\tau)}{d \ln \tau} \right],$$

$$u = \tau(r_1(\tau))^{-1/n} w^{1/n} + (1-\tau)(r_2(\tau))^{-1/n} z^{1/n} e^{it},$$

$$\Phi = F + n w F_w' + n z F_z',$$

the author proves the integral formulas [cf. the preceding review]

$$F(w, z) = (4\pi^2 i)^{-1} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|c|=1} (\zeta - u)^{-2} \zeta F(r_1(\tau) \zeta^n, \\ \times r_2(\tau) (\zeta e^{-it})^n) d\zeta, \\ F(w, z) = (4\pi^2 i)^{-1} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|c|=1} (\zeta - u)^{-1} \Phi(r_1(\tau) \zeta^n, \\ \times r_2(\tau) (\zeta e^{-it})^n) d\zeta.$$

H. Tornehave (Copenhagen)

5885:

Kruming, A. A. An analog of the interior area theorem for pseudoconformal mapping of the unit bicylinder. *Moskov. Oblast. Pedagog. Inst. Uč. Zap.* 57 (1957), 39-42. (Russian)

The author considers a pair of analytic functions f and g in the unit bicylinder $|w| < 1$, $|z| < 1$. He computes the fourdimensional volume of the image of $|w| \leq r < 1$, $|z| \leq \rho < 1$ by the mapping (f, g) expressed as a power series in r and ρ and he studies the behaviour of this power series when $(r, \rho) \rightarrow (1, 1)$.

H. Tornehave (Copenhagen)

5886:

Bavrin, I. L. The nature of a pair of analytic functions, one of which is entire, which are univalent in the space of two complex variables. *Moskov. Oblast. Pedagog. Inst. Uč. Zap.* 57 (1957), 33-37. (Russian)

The author proves that a one-to-one mapping of the complex (w, z) -plane into itself by a pair of analytic functions $W = W(w, z)$, $Z = Z(w, z)$, of which W is entire, has the form

$$W = az + b,$$

$$Z = (\alpha(W)w + \beta(W))(\gamma(W)w + \delta(W))^{-1},$$

or the form obtained from this by interchanging w and z . The functions α , β , γ , δ are entire, the determinant $\alpha\delta - \beta\gamma$ is $\neq 0$, a and b are constants and $a \neq 0$.

H. Tornehave (Copenhagen)

5887:

Kruming, A. A. Univalent mappings of the unit bicylinder. *Moskov. Oblast. Pedagog. Inst. Uč. Zap.* 57 (1957), 29-32. (Russian)

The author considers a one-to-one mapping $W = f(w, z)$, $Z = \varphi(w, z)$ of the unit bicylinder by a pair of analytic functions with $f'_w \neq 0$. The equation $f(w, z) = C$ is solved in the form $w = \psi(z, C)$, and the derivative of $\Phi(z, C) = \varphi(\psi(z, C), z)$ is estimated. The author proves that Φ is one-to-one for $|z| < 1$, but he has not taken into account that Φ need not be defined everywhere in the unit circle, when C is a fixed value assumed by $f(w, z)$. The author ends up with some estimates like the following

$$(1+R)^{-4}(1-R)^4 \leq \left| \frac{f'_z(w, z)f'_w(0, 0)}{f'_w(w, z)f'_z(0, 0)} \right| \leq (1-R)^{-4}(1+R)^4,$$

where $R^2 = |w|^2 + |z|^2$, but the conditions under which this holds are not clearly stated. H. Tornehave (Copenhagen)

5888:

Piatetsky-Shapiro, I. I. Estimate of the dimensionality of the space of automorphisms for certain types of discrete groups. *Dokl. Akad. Nauk SSSR (N.S.)* 113 (1957), 980-983. (Russian)

The fundamental theorem of the present note is as follows: "Let D be a bounded region in N -dimensional complex space and let Γ be a discrete group of analytic automorphisms of D of regular structure. The dimension-

ality of the space of automorphic forms of weight m is not greater than cm^N , where c does not depend on m but only on D and Γ ."

5889:

Cartan, Henri. Fonctions automorphes et séries de Poincaré. *J. Analyse Math.* 6 (1958), 169-175.

The author has given theorems in his seminar lectures on the theory of automorphic functions in analytic spaces (1953-1954) concerning the existence of Poincaré series, admitting developments within limits in given points. He now states that the demonstration of these theorems was incorrect, whilst the theorems themselves are correct. They have already found applications in various cases. The demonstration as stated here starts from two lemmas and concerns three theorems. A consequence of the first theorem is pointed out. Reasoning is straightforward and starts (lemma 1) from a scalar bounded holomorphic function. M. J. O. Strutt (Zurich)

5890:

Baily, Walter L., Jr. Satake's compactification of V_n . *Amer. J. Math.* 80 (1958), 348-364.

Soit H_n le demi-plan généralisé de Siegel (espace des matrices complexes symétriques à n lignes et n colonnes, dont la partie imaginaire est définie positive); soit M_n le groupe modulaire $Sp(n, Z)$ qui opère dans H_n . L'espace $V_n = H_n/M_n$ est un espace analytique normal; il a été compactifié par I. Satake [*J. Indian Math. Soc. (N.S.)* 20 (1956), 259-281; MR 18, 934]. Le compactifié

$$V_n^* = V_n \cup V_{n-1} \cup \dots \cup V_1 \cup V_0$$

est muni par Satake d'une structure d'espace annelé, dont il faut démontrer que c'est bien une structure d'espace analytique normal. L'auteur le prouve en établissant d'abord un théorème général de prolongement des espaces analytiques normaux (th. 1); ce théorème de prolongement a été ultérieurement amélioré par le rapporteur [voir l'analyse ci-dessous]. Il s'applique ici grâce aux propriétés des formes automorphes d'un poids donné assez grand.

L'auteur démontre ensuite que V_n^* est isomorphe (comme espace analytique compact) à une variété algébrique projective, projectivement normale; il en résulte que, pour $n \geq 2$, toute fonction méromorphe invariante par M_n est le quotient de deux formes automorphes de même poids. L'auteur annonce des résultats analogues dans le cas où le groupe M_n est remplacé par un sous-groupe d'indice fini de M_n . Signalons que, pour tout groupe M' commensurable à M_n , des résultats plus précis relatifs au plongement projectif du compactifié de H_n/M' ont été obtenus postérieurement par I. Satake et le rapporteur [*Séminaire H. Cartan*, 10e année: 1957/58, Secrétariat Math., Paris, 1958; cf. notamment exposé 17].

H. Cartan (Paris)

5891:

Cartan, Henri. Prolongement des espaces analytiques normaux. *Math. Ann.* 136 (1958), 97-110.

Let X be a locally compact space; let V be an open subset which is everywhere dense, and let $W = X - V$. Suppose V is a normal complex analytic space of dimension m . The problem of analytic prolongation is to define on X a structure of normal analytic space in such a way that: (α) The structure of V is that induced from X ; (β) W is an analytic subspace of dimension $< m$.

It is easily seen that there cannot be more than one

structure of analytic space on X which satisfies (α) and (β) , namely, for each $x \in X$ we define the stalk A_x of germs of holomorphic functions at x as the stalk of germs of continuous functions which belong to B_y for $y \in V$ sufficiently close to x . (Here B is the sheaf defining the analytic structure on V .)

The main result of the paper is Theorem 2: Suppose (i) Each $x_0 \in W$ has (in X) a fundamental system of open neighborhoods U such that $V \cap U$ is connected; (ii) every $x_0 \in W$ has a neighborhood U such that the functions on U which are continuous and holomorphic on $V \cap U$ separate the points of $V \cap U$; (iii) A defines on W the structure of analytic space of dimension $< m$; then A defines the structure of analytic normal space on X .

The origin of this problem stems from the Satake compactification of the fundamental domain for the symplectic modular group in Siegel's upper half plane. Baily [see the preceding review] has proved that this compactification is a normal analytic space; Baily's theorem is an easy consequence of the results of this paper.

L. Ehrenpreis (Waltham, Mass.)

SPECIAL FUNCTIONS

See also 5871, 5919.

5892:

Balk, M. B. A property of the Bernoulli numbers. *Moskov. Oblast. Pedagog. Inst. Uč. Zap.* 57 (1957), 55-59. (Russian)

The present note is concerned with an arithmetical property of the Bernoulli numbers. For any positive integer m , denote by δ_m the value of the m -rowed determinant whose (k, j) th element is equal to $B_{2k+2j-2}/(2k+2j-2)!$. Then we have

$$\delta_m^{-1} = x_m^2 \cdot 3 \cdot 7 \cdot 11 \cdots (4m-1),$$

where x_m is an even integer. The proof depends on the Euler-Lambert representation of $\tan z$ as a continued fraction and on the relation between power series and certain continued fractions associated with them [O. Perron, *Die Lehre von den Kettenbrüchen*, 2nd ed., Teubner, Leipzig-Berlin, 1929; p. 304]. L. Mirsky (Sheffield)

5893:

Hancock, Harris. *Elliptic integrals*. Dover Publications, Inc., New York, 1958. 104 pp. \$1.25.

An unaltered (paper-covered) reproduction of the original edition [Wiley, New York, 1917].

5894:

Gelfer, S. A. On the maximum to the conformal radius of the fundamental region of a doubly-periodic group. *Dokl. Akad. Nauk SSSR (N.S.)* 114 (1957), 241-244. (Russian)

Let $\{D\}$ be a family of simply connected regions D of the plane w containing the point $w=0$ with the following properties: 1) the region D does not contain points which are congruent with respect to the group T_n of transformations

$$w' = w + n_1 \omega_1 + n_2 \omega_2,$$

where ω_1 and ω_2 are constants whose ratio is not purely real and n_1 and n_2 are arbitrary integers; 2) the regions

D do not contain a given system of finite points a_1, \dots, a_m nor points congruent to them with respect to T_n .

The article studies the problem of finding amongst all regions in $\{D\}$ that region which has the greatest conformal radius.

From the introduction

5895:

Fel'dman, N. I. Joint approximations of the periods of an elliptic function by algebraic numbers. *Izv. Akad. Nauk SSSR Ser. Mat.* 22 (1958), 563-576. (Russian)

"This article introduces an estimate from below for the sum $|\omega - \xi| + |\omega_1 - \xi_1|$, where ω and ω_1 are the periods of the function $\wp(z)$ with algebraic invariants and ξ, ξ_1 are algebraic numbers. The estimate depends on the degree and height of the numbers ξ, ξ_1 and on the degree of the field formed by the adjunction of ξ and ξ_1 to the rational field."

Author's summary

5896:

Bihari, Imre. On a monotonic property of Bessel functions. *Mat. Lapok* 7 (1956), 43-46. (Hungarian. Russian and English summaries)

Makai gave a new approach to an earlier result of R. G. Cooke stating that the "waves" of the Bessel function $J_\nu(x)$ are decreasing for $x > 0$ provided $\nu > -1$. Makai's proof yields this for $|\nu| > \frac{1}{2}$. By a simple modification of Makai's argument the present paper succeeds in settling the case $\frac{1}{2} \leq \nu < \frac{3}{2}$.

G. Szegő (Stanford, Calif.)

5897:

Wintner, Aurel. A note on Mathieu's functions. *Quart. J. Math. Oxford Ser. (2)* 8 (1957), 143-145.

From a general theorem on completely monotone solutions of ordinary linear differential equations, it is concluded that Mathieu's differential equation

$$\frac{d^2 y}{d\varphi^2} + (p - 2q \cos 2\varphi)y = 0$$

with $p \geq 0, q > 0$ has a solution of the form

$$y(\varphi) = \int_0^\infty \exp(-t \cos \varphi) \cos(t \sin \varphi) dF(t)$$

in $-\frac{1}{2}\pi < \varphi < \frac{1}{2}\pi$, where $F(t)$ ($\neq 0$) is a non-decreasing function of t in $0 \leq t < \infty$. Nothing is yet known as to the explicit form of $F(t)$.

J. Meixner (Aachen)

5898:

Chaundy, T. W. On Clausen's hypergeometric identity. *Quart. J. Math. Oxford Ser. (2)* 9 (1958), 265-274.

This is a virtuosic set of variations on the theme

$$[F(a, b; a+b+\frac{1}{2}; x)]^2 =$$

$${}_3F_2(2a, a+b, 2b; a+b+\frac{1}{2}, 2a+2b; x)$$

by Clausen. First, Clausen's formula is proved by considering the differential equations satisfied by each of the two sides. This leads to a companion formula for the product of two solutions of the hypergeometric equation when $c = a+b+\frac{1}{2}$. Clausen's identity is then extended to unrestricted c , and comparison of the coefficients on both sides of the extended identity lead to transformation formulas for terminating ${}_4F_3$ of unit argument. Next, an expansion of $[F(a, b; c; x)]^2$ is obtained when $c = a+b+\frac{1}{2}$, and, finally, a much more difficult one when $c = a+b+\frac{1}{2}$.

The paper contains contributions by J. L. Burchall, to the handling of technical difficulties in manipulating the differential equations. A. Erdélyi (Pasadena, Calif.)

5899:

Al-Salam, Waleed A. On a characterization of orthogonality. *Math. Mag.* 31 (1957/58), 41-44.

A result of D. Dickinson [*Proc. Amer. Math. Soc.* 5 (1954), 946-956; MR 19, 263] is generalized as follows. Given two sequences of polynomials $\{f_n(x)\}$, $\{g_n(x)\}$ of degrees n and $n-1$, respectively, $f_0 \neq 0$, $g_0 = 0$, $g_1(x) \neq 0$; the necessary and sufficient condition that a relation of the form

$$f_n(x)g_{n+1}(x) - f_{n+1}(x)g_n(x) = a_n \neq 0$$

exists is that both $\{f_n\}$ and $\{g_n\}$ satisfy a recurrence of type

$$u_{n+1}(x) = (A_n x + B_n)u_n(x) + C_n u_{n-1}(x); C_n \neq 0,$$

$$n = 1, 2, \dots$$

G. Szegő (Stanford, Calif.)

5900:

Al-Salam, W. A. On the Turan inequality for certain polynomials. *Amer. Math. Monthly* 66 (1959), 46-49.

The polynomials $p_n(x)$ are defined by the recursion

$$p_{n+1}(x) = A_n x p_n(x) - C_n p_{n-1}(x), n \geq 1,$$

where $p_0(x) = 1$, $p_1(x) = A_0 x$; moreover $A_n > 0$, $C_n > 0$, $A_n \uparrow$, $C_n/A_n \uparrow$. Then for any integer k and for x real, the derivatives $p_n^{(k)}(x) = p_n^{(k)}$ satisfy the inequality

$$p_n^{(k)} - p_{n-1}^{(k)} p_{n+1}^{(k)} \geq (k! \prod_{r=0}^{k-1} A_r)^2 \cdot \prod_{r=k}^{n-1} C_r \cdot A_n / A_k.$$

G. Szegő (Stanford, Calif.)

5901:

Rajagopal, A. K. A note on the unification of the classical orthogonal polynomials. *Proc. Nat. Inst. Sci. India. Part A* 24 (1958), 309-313.

"Theorem (a): The condition that the polynomials defined by

$$p_m(n, x) = \frac{1}{k_m(w(x))} \left(\frac{d}{dx} \right)^m (w(x) X^n)$$

((m, n) are positive integers) may have a second order differential equation is $X^{(m)} + (nX' + (w'/w)X)'' = 0$. $\dots X$ is a polynomial of degree k in $x \dots \log w = -\int^x h(t)/X(t) dt$ where $h(t)$ is a polynomial of degree $(k-1)$." {The meaning and the proof of this theorem are not clear to the reviewer.} The author proceeds to show that for $m=n$, $k=2$ one obtains the classical orthogonal polynomials [see also Tricomi, *Vorlesungen über Orthogonalreihen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955; MR 17, 30; p. 129f.] and writes down two non-orthogonal polynomials for $k=3$, together with their differential equations. A. Erdélyi (Pasadena, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 5835, 6043, 6229, 6249, 6250, 6251, 6328.

5902:

Hukuhara, Masuo. Sur la théorie des équations différentielles ordinaires. *J. Fac. Sci. Univ. Tokyo. Sect. I.* 7 (1958), 483-510.

Consider a system of differential equations

$$\frac{dy_j}{dx} = f_j(x, y_1, \dots, y_n) \quad (j=1, \dots, n),$$

or briefly, $dy/dx = f(x, y)$, and suppose at first that f is continuous in its domain of existence D .

The author defines a right neighborhood of a point (ξ, η) of D to be the set $V_\varepsilon^+(\xi, \eta)$ of points (x, y) such that

$$0 \leq x - \xi \leq \varepsilon, |y - \eta - (x - \xi)f(\xi, \eta)| \leq \varepsilon(x - \xi),$$

where ε is an arbitrary positive number. The right neighborhoods of points not in D are simply the points themselves. The system of right neighborhoods defines a right topology. Similarly, there are left neighborhoods and a left topology relative to the differential system. The foregoing system of differential equations is said to be of Carathéodory type provided: (1) $f(x, y)$, considered as a function of y , is continuous in D , and (2) if $y(x)$ is a continuous function and $f(x, y(x))$ is defined almost everywhere in the interval in which $y(x)$ is defined, then the function $f(x, y)$ is measurable.

The notions of right and left topologies are extended to differential systems of Carathéodory type and of other types, and through these notions a unified systematic theory of differential systems of the types considered is developed. E. F. Beckenbach (Los Angeles, Calif.)

5903:

Andreev, A. F. On the first problem of distinguishing in Frommer's theory. *Vestnik Leningrad. Univ.* 13 (1958), no. 13, 84-86. (Russian. English summary)

Consider

$$\frac{dy}{dx} = \frac{Q(x, y) + q(x, y)}{P(x, y) + p(x, y)},$$

where P, Q are homogeneous polynomials of degree $n \geq 1$ and p, q are continuous and sufficiently smooth functions, $p = O(r^n)$, $q = O(r^n)$. Let P^*, \dots be the functions $P(r \cos \varphi, r \sin \varphi), \dots$,

$$F + iG = r^{-n}(Q^* + iP^*)e^{i\varphi}$$

$$f + ig = r^{-n}(q^* + ip^*)e^{i\varphi}$$

and suppose that $F^{(k)}(0) = 0$, $0 \leq k \leq n-1$, $F^{(k)}(0)G(0) < 0$ for some odd integer $k \geq 1$. Then the polar equation

$$r \frac{d\varphi}{dr} = \frac{F(\varphi) + f(r, \varphi)}{G(\varphi) + g(r, \varphi)}$$

has in the sector $S: 0 \leq r < R, |\varphi| \leq \Delta$ (R, Δ sufficiently small) either a unique solution $\varphi = \varphi(r)$ such that $\varphi \rightarrow 0$ with r or infinitely many such solutions. The author states without proof a theorem which implies the uniqueness if, in addition, $r^{-n}|p| \leq Kc(r)$, $r^{-n}|q| \leq Kc(r)$, where $c(r)$ is a continuous function such that $\int_0^1 r^{-1}c(r)dr < \infty$, and f, g are in S Lipschitzian in φ for $c(r)$. This improves on results of Lonn [*Math. Z.* 44 (1938), 506-530] and Haimov [*Uč. Zap. Stalin. Ped. Inst.* 1 (1952)]. For related results see Lefschetz [Contributions to the theory of non-linear oscillations, vol. 2, Princeton Univ. Press, 1952; p. 61-73; *Bol. Soc. Mat. Mexicana* (2) 1 (1956), 13-27; MR 14, 557; 18, 481].

H. A. Antosiewicz (Los Angeles, Calif.)

5904:

Grobman, D. M. Exponents and minus-exponents of systems of ordinary differential equations. *Mat. Sb. N. S.* 46(88) (1958), 343-358. (Russian)

The author introduces the concept of a minus-exponent of a solution of a differential equation by means of the expression

$$\limsup_{t \rightarrow \infty} \left(-\log \left(\sum_{i=1}^N |y_i(t)| \right) \right) / t,$$

and then establishes some results pertaining to these and the usual exponents. R. Bellman (Santa Monica, Calif.)

5905:

Plis, A.; and Ważewski, T. A uniqueness condition with a standard differential equation without uniqueness property. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 6 (1958), 145-148.

In the system (1) $X' = F(t, X)$ a solution $X = X(t)$ ($\alpha < t < \beta$) will be called unique to the right if, for any solution $Y(t)$ of (1) defined in $\gamma \leq t \leq \delta$, $\alpha < \gamma < \delta < \beta$, and satisfying $Y(\gamma) = X(\gamma)$, we have $Y(t) = X(t)$, $\gamma \leq t \leq \delta$. The main theorem also requires the following definition: We say that a function $g(t, x)$ defined in a subset S of the half-plane $x > 0$ has property P if for every real number s there exists a sequence $x_n(t)$ of positive solutions of $x' = g(t, x)$ such that either (I) there exists $\epsilon > 0$ such that $\lim_{n \rightarrow \infty} x_n(t) = 0$ for $s \leq t \leq s + \epsilon$, or (II) there exists a sequence of numbers $t_n > s$, $t_n \rightarrow s$ as $n \rightarrow \infty$, and for each fixed n , $x_n(t) \rightarrow 0$ as $t \rightarrow t_n$ ($s < t < t_n$).

Theorem: Suppose that the right hand members of the system (1), defined in the open set R , satisfy the inequality

$$|F(t, X) - F(t, Y)| < g(t, |X - Y|)$$

for $(t, X) \in R$, $(t, Y) \in R$, $(t, X - Y) \in S$, $|A| = (a_1^2 + a_2^2 + \dots + a_n^2)^{1/2}$, where the function $g(t, x)$ has property P . Then every solution of (1) is unique to the right.

R. R. Kemp (Kingston, Ont.)

5906:

Hustý, Zdeněk. Über einige Eigenschaften der Picardsfolgen. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 8 (1958), 7-19. (Czech. Russian and German summaries)

Let $\{y_k(x)\}_{k=0}^{\infty}$ be the Picard successive approximations for the differential equation (1) $y' = f(x, y)$ and let either $y_0' \geq f(x, y_{n-1}(x))$ or $y_0' \leq f(x, y_{n-1}(x))$ hold. If the function $f(x, y)$ is monotone in y , then the sequences $\{y_{n+s}\}_{s=0}^{\infty}$ ($s=0, 1, \dots, n-1$) converge uniformly and their limits Y_s are the solutions of the system (2) $Y_s' = f(x, Y_{s-1})$, $s=1, \dots, n-1$, $Y_0' = f(x_1, Y_{n-1})$. In the case that $f(x, y)$ is nondecreasing in y , the system (2) has only the trivial solution $Y_0 = Y_1 = \dots = Y_{n-1} = y$, where $y(x)$ is the solution of (1). If $f(x, y)$ is nonincreasing in y , then the system (2) can also have solutions other than that trivial one. The case $n=2$ is studied more in detail.

M. Zlámál (Brno)

5907:

Rybarski, A. Über eine gewisse Linearisationsmethode der Differentialgleichungen vom Pendeltypus. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 6 (1958), 175-179.

Let y_0 be a strictly monotonic, continuous solution of

$$(1) \quad y'' + g(y) = 0, \quad g(y) \in C(-\infty, \infty),$$

on $[0, T]$, except possibly at a finite set of points t_1, \dots, t_n , where y_0'' and y_0' may have finite jumps. The problem dealt with in this paper is the approximation of y_0 by the solution $y_{ap}(t, a, b)$ of a linear equation

$$(2) \quad y'' + ay + b = 0,$$

under the condition

$$(3) \quad y_{ap}(0, a, b) = y_0(0); \quad y_{ap}(T, a, b) = y_0(T);$$

$$y_{ap}(t_k, a, b) = y_0(t_k) \quad (k=1, \dots, n).$$

It is assumed that $0 \leq a < w^2$, where $w = \pi / \max_{0 \leq k \leq n} |t_{k+1} - t_k|$ ($t_0 = 0, t_{n+1} = T$). Under this condition the boundary value problem (2) and (3) is solvable. It is then shown that

$$(4) \quad \|y_0' - y_{ap}'\| \leq w \|g(y_0) - ay_0 - b\| / (w^2 - a),$$

where the norm is that of $L_2(0, T)$. The "best linear

approximation" to equation (1) is determined, in the sense that the constants $a \in [0, w^2)$ and $b \in (-\infty, \infty)$ which minimize the right side of (4) are found. In certain cases, the "best approximation" can be replaced by one which differs little from it, but is more practical for computational purposes. An example is given in the last section. There is a misprint in the formula for y_{ap} in this example; under the given boundary conditions, the solution should be

$$(5) \quad y_{ap}(t) = x_m \{ \sin at + \sin a(t-T) \} / \sin aT.$$

J. Elliott (New York, N.Y.)

5908:

Sibirskii, K. S. The principle of symmetry and the problem of the center. *Kišinev. Gos. Univ. Uč. Zap.* 17 (1955), 27-34. (Russian)

A generalization of the results of same Uč. Zap. 11 (1954), 115-117 [MR 17, 737] to the equation $dy/dx = -Y(x, y)/X(x, y)$, where the functions X and Y are analytic in a neighborhood of the origin and vanish at the origin.

5909:

Volkov, I. F. Solution of a linear homogeneous differential equation of n th order with constant real coefficients. *Kišinev. Gos. Ped. Inst. Uč. Zap.* 3 (1955), 17-21. (Russian)

A methodological note, pointing out the possibility of obtaining the general solution of the equation of the title without recourse to the theory of functions of a complex variable. B. P. Demidovič (RŽMat 1956 #1311)

5910:

Wintner, Aurel. On the integration of ordinary linear differential equations of second order by means of Laplace transforms. *Rev. Mat. Hisp.-Amer.* (4) 18 (1958), 71-80.

A function $f(x)$ of class C^∞ on $(0, \infty)$ is called completely monotone (c.m.) if $(-1)^n D^n f \geq 0$ for $n=0, 1, \dots$, where $D = d/dx$, that is, if and only if (Hausdorff-Bernstein) $f(x)$ is representable in the form $f(x) = \int_0^\infty e^{-xs} d\mu(s)$, $d\mu \geq 0$ on $0 \leq s < \infty$. The author proved [Amer. J. Math. 69 (1947), 87-98; MR 8, 381] that if f is c.m., then up to positive constant factors) $D^2 y - f(x)y = 0$ has a (unique) c.m. solution $y = y(x)$. The first part of this paper gives several consequences of this fact depending on changes of the independent variable and on variations of constants.

Let $D^* = t^{-1}d/dt$, so that $g(t)$ of class C^∞ on $(0, \infty)$ is representable in the form (1) $g(t) = \int_0^\infty \exp(-t^2 s/2) d\mu(s)$, $d\mu \geq 0$ for $0 \leq s < \infty$, if and only if $(-1)^n D^{*n} g \geq 0$ for $n=0, 1, \dots$. In the second part of the paper it is pointed out that this fact leads at once to P. Lévy's theorem that $g(t) = \exp(-t^\alpha)$, $0 \leq \alpha \leq 2$, is representable in the form (2) $g(t) = \int_0^\infty \cos ts dv(s)$, $dv \geq 0$ on $0 \leq s < \infty$ [cf. Kuttner, J. London Math. Soc. 19 (1944), 77-84; MR 7, 59] and also to the known result that $g(t)$ is representable in the form (1). P. Hartman (Baltimore, Md.)

5911:

Vinograd, R. È. Conjugate Lyapunov norms. *Dokl. Akad. Nauk SSSR (N.S.)* 119 (1958), 415-417. (Russian)

The author introduces conjugate Lyapunov norms [cf. Bogdanov, same Dokl. 113 (1957), 255-257; MR 19, 754] for which he states (without proof) a number of theorems. As an illustration he cites the Perron order numbers of a system $\dot{x} = A(t)x$ and of its adjoint $\dot{z} = -A^*(t)z$.

H. A. Antosiewicz (Los Angeles, Calif.)

5912:

Sahnovič, L. A. Inverse problem for differential operators of order $n > 2$ with analytic coefficients. *Mat. Sb. N. S.* 46(88) (1958), 61-76. (Russian)

Consider the differential equation of arbitrary order $n > 2$,

$$(1) \quad \frac{d^n y}{dx^n} + q(x)y - \lambda^n y = f(x) \quad (0 \leq x \leq l),$$

with the initial conditions

$$(2) \quad y(0, \lambda) = 1, \quad \left. \frac{d^k y(x, \lambda)}{dx^k} \right|_{x=0} = 0 \quad (k = 1, 2, \dots, n-1).$$

The author shows that if $q(x)$ is analytic in a circle of suitable radius R , then the solution of (1) may be written in the form

$$(3) \quad y(x, \lambda) = \omega(x, \lambda) + \int_0^x \omega(t, \lambda) K(x, t) dt,$$

where $\omega(x, \lambda)$ is the solution of

$$(4) \quad \frac{d^n \omega}{dx^n} - \lambda^n \omega = 0, \quad \omega(0, \lambda) = 1, \quad \left. \frac{d^k \omega}{dx^k} \right|_{x=0} = 0 \quad (k = 1, 2, \dots, n-1),$$

and $K(x, t)$ has continuous partial derivatives to order n inclusive and satisfies the relations

$$\frac{\partial^n}{\partial x^n} K(x, t) - (-1)^n \frac{\partial^n}{\partial t^n} K(x, t) + q(x)K(x, t) = 0,$$

$$\left. \frac{\partial^k}{\partial x^k} K(x, t) \right|_{t=x} = 0 \quad (k = 0, 1, 2, \dots, n-3),$$

$$\left. \frac{\partial^{n-2}}{\partial x^{n-2}} K(x, t) \right|_{t=x} = -\frac{1}{n} \int_0^x q(t) dt,$$

$$(-1)^{n+1} \left. \frac{\partial^{n-1}}{\partial t^{n-1}} K(x, t) \right|_{t=0} = f(x) \quad (0 \leq x \leq l).$$

A formula of type (3) was given for $n=2$ by A. Ya. Povzner [*Mat. Sb.* 23(65) (1948), 3-52; MR 10, 299] and generalized for $n > 2$ by M. K. Fage [*Dokl. Akad. Nauk SSSR* 108 (1956), 1022-1025; MR 18, 580]. The present treatment is separate from Fage's and yields a simpler formulation. *J. F. Heyda* (Cincinnati, Ohio)

5913:

Reid, William T. A Prüfer transformation for differential systems. *Pacific J. Math.* 8 (1958), 575-584.

Consider the matrix differential system (1) $Y' = G(x)Z$, $Z' = -F(x)Y$, $a \leq x < \infty$, where $F(x)$, $G(x)$ are continuous $n \times n$ hermitian matrices. Let $Y(x)$, $Z(x)$ be of rank n , satisfy (1) and be such that $Y^*(x)Z(x) - Z^*(x)Y(x) = 0$ where $*$ denotes conjugate transpose. Then there exists a continuously differentiable non-singular matrix $R(x)$ and a continuous hermitian matrix $Q(x)$ for which the system $\phi' = Q(x)\psi$, $\psi' = -Q(x)\phi$ has a solution $\phi(x)$, $\psi(x)$ such that $\phi(x)\phi^*(x) + \psi(x)\psi^*(x)$ is the identity matrix and $Y(x) = \phi^*(x)R(x)$, $Z(x) = \psi^*(x)R(x)$. Furthermore, if the above result is true for $R(x) = R_1(x)$, $Q(x) = Q_1(x)$ then the most general form of these matrices is $R(x) = \Gamma R_1(x)$, $Q(x) = \Gamma Q_1(x)$, where Γ is constant and unitary. This is the main result of the paper, which generalizes a theorem of J. H. Barret [*Proc. Amer. Math. Soc.* 8 (1957), 510-518; MR 19, 415] corresponding to the case where $Y(a) = 0$, $G(x)$ and $F(x)$ are real and $G(x)$ is non-singular. The paper also improves and extends some oscillation theorems of Barret. *M. M. Peixoto* (Baltimore, Md.)

5914:

Dias Agudo, F. R. On non symmetric linear differential operators of the second order. *Univ. Lisboa. Revista Fac. Ci. A* (2) 6 (1957/58), 177-190.

Let

$$\tau = \tau_1 + i\tau_2 = -\frac{d}{dx} \left(p \frac{d}{dx} \right) + q_1 + iq_2,$$

where $p > 0$ and $q_2 \geq 0$ on an interval (open, closed, half-open) of the real line. The author uses the abstract theory of linear operators to prove a number of results concerning differential operators of the type τ , and to some extent concerning generalizations to the higher order case. Certain theorems analogous to those occurring in the Weyl limit-circle and limit-point theory of singular boundary value problems are proved.

C. R. Putnam (Lafayette, Ind.)

5915:

Rasulov, M. L. A formula for the expansion of an arbitrary function. *Dokl. Akad. Nauk SSSR (N.S.)* 119 (1958), 450-453. (Russian)

The differential system

$$\frac{dy^{(i)}}{dx} - A^{(i)}(x, \lambda)y^{(i)} = f^{(i)}(x),$$

$$a_{kj}^{(i)}(x, \lambda) = \lambda a_{kj}^{(i)} + \sum_0^N \lambda^{-r} a_{kj}^{(i)}(x) \quad (i = 1, \dots, m),$$

is considered on the interval $[a_0, a_m]$, $a_0 < a_1 < \dots < a_m$, with boundary conditions

$$\sum_{i=1}^m \{ \alpha_{kj}^{(i)}(\lambda) y_j^{(i)}(a_{i-1}) + \beta_{kj}^{(i)}(\lambda) y_j^{(i)}(a_i) \} = 0.$$

The elements of the matrix $A^{(i)}(x, \lambda)$ are assumed to be twice continuously differentiable functions of x on (a_{i-1}, a_i) having finite jumps at the points a_i . The author has previously considered a problem for n th order equations [*Izv. Akad. Nauk Azerbaidžan. SSR* 1953, no. 6, 3-28; MR 17, 740]. The author announces that under certain restrictions on the characteristic roots (including that they be distinct and non-zero), on the rank of the matrix $(\alpha_{kj}^{(i)}, \beta_{kj}^{(i)})$, and on $\Delta(\lambda)$, the following result can be obtained: There exists a sequence of closed expanding contours Γ_ν ($\nu = 1, 2, \dots$) such that for each vector function $f^{(i)}(x)$ with components $f_k^{(i)}(x) \in L_2(a_{i-1}, a_i)$ the integral

$$\frac{i}{2\pi} \int_{\Gamma_\nu} y^{(i)}(x, \lambda) d\lambda$$

converges in the mean to $[A^{(i)}(x)]^{-1} f^{(i)}(x)$, when $A^{(i)}(x) = (a_{kj}^{(i)}(x))$. The methods used are those of Birkhoff [*Trans. Amer. Math. Soc.* 9 (1908), 214-231] and Tamarkin [*Math. Z.* 27 (1928), 1-54].

N. D. Kazarinoff (Ann Arbor, Mich.)

5916:

Feščenko, S. F.; and Škil', M. I. On the asymptotic solution of a special system of ordinary linear differential equations. *Dopovidi Akad. Nauk Ukrain. RSR* 1958, 482-485. (Ukrainian. Russian and English summaries)

In this paper the authors present two theorems which establish the asymptotic solution of the system of differential equations:

$$(1) \quad \frac{dX}{dt} = [A^0(\tau) + \varepsilon A^1(\tau)]X + B(\tau)e^{i\theta(\tau, \varepsilon)},$$

where $A^0(\tau)$ and $A^1(\tau)$ are certain square matrices of order n , X and $B(\tau)$ are n -space vectors, and ε is a small positive parameter. *S. Kulik* (Logan, Utah)

5917:

Ráb, Miloš. Über die Differentialgleichung $y''' + 2A(x)y' + [A'(x) + \omega(x)]y = 0$. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 8 (1958), 115-122. (Czech. Russian and German summaries)

The asymptotic behavior of the solutions of the third order linear differential equation in the canonical form

$$(1) \quad y''' + 2A(x)y' + [A'(x) + \omega(x)]y = 0$$

is studied here. Under certain assumptions the author proves that the solutions of (1) behave, in certain respects, asymptotically the same as in the case of the constant coefficients; e.g., the equation (1) has also nonoscillatory solutions and every such solution diverges monotonously to $\pm\infty$, every oscillatory solution is of the class $L^2(x_0, \infty)$, and the zeros of each two independent oscillatory solutions separate each other. M. Zlámal (Brno)

5918:

Višik, M. I.; and Lyusternik, L. A. On the asymptotic behaviour of the solutions of boundary problems for quasilinear differential equations. Dokl. Akad. Nauk SSSR 121 (1958), 778-781. (Russian)

The method developed by the authors for the asymptotic solution of linear differential equations involving a small parameter [same Dokl. 113 (1957), 734-737; Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 3-122; MR 19, 861; 20#2539] can be applied successfully to nonlinear equations of the form $\epsilon y'' + f(x, y, y') = 0$ as well. This application is specifically developed only for the equation $\epsilon y'' + \varphi(x, y)y' - \psi(x, y) = 0$, with boundary conditions $y(0) = A$, $y(1) = B$, under the assumption that $\varphi(x, y) \geq b > 0$. This problem and somewhat more general ones have been treated by Wasow [Comm. Pure Appl. Math. 9 (1956), 93-113; MR 18, 39].

N. D. Kazarinoff (Ann Arbor, Mich.)

5919:

Kazarinoff, Nicholas D. Asymptotic theory of second order differential equations with two simple turning points. Arch. Rational Mech. Anal. 2 (1958), 129-150.

The author considers second order ordinary linear differential equations of the form

$$(1) \quad d^2y/ds^2 - \lambda^2 P(s, \lambda)y = 0,$$

where $P(s, \lambda) = \sum_{j=0}^{\infty} p_j(s)\lambda^{-j}$ is analytic in s in a region D of the complex plane s and in λ for $|\lambda| > N$, and where $p_0(s)$ has exactly two simple zeros α, β in the interior of D (two turning points in the terminology now well known after the work of Langer). As in Langer's theory, which has a great bearing in the present paper, the problem is to find asymptotic expansions for the solutions of (1) uniformly valid in D for $|\lambda| \rightarrow +\infty$. The equation (1) is transformed first into an analogous equation of the form

$$(2) \quad d^2u/dz^2 - \lambda^2 Q(z, \lambda)u = 0,$$

where $Q(z, \lambda) = \sum_{j=0}^{\infty} q_j(z)\lambda^{-j}$, $q_0(z) = z^2 - 1$, and $Q(z, \lambda)$ is analytic in a region R containing the two turning points $z = \pm 1$ in its interior. The main point consists in taking as "leading terms" of the asymptotic expansion for (2), or "first approximation", the solutions of the Weber equation

$$d^2V/dt^2 + [v/2 - t^2/4]V = 0,$$

which is here transformed into

$$d^2v/dt^2 - \lambda^2[(z^2 - 1) + \lambda^{-1}c + \dots]v = 0,$$

by a convenient change of variables and choice of the

parameter. Finally, as in Langer's theory, successive approximations are defined, the n th approximation satisfying an equation of the same form, where the expression in brackets coincides with $Q(z, \lambda)$ up to the terms of degree $n-1$ in λ . Previous results of Langer and McKelvey are used. It is then proved that the n th approximation is a valid asymptotic expansion to $n+1$ terms of solutions of (2). Specific hypotheses are made during the process which cannot be referred to here. The behavior of the solutions of equation (2) are known inasmuch as one knows the behavior of the Weber functions, which are the leading terms of the expansion. In particular, results of Erdélyi, Kennedy and McGregor are used. An algorithm is given to determine recursively the terms of the asymptotic expansions of these Weber functions.

L. Cesari (Baltimore, Md.)

5920:

Urabe, Minoru. Moving orthonormal system along a closed path of an autonomous system. J. Sci. Hiroshima Univ. Ser. A 21 (1957/58), 177-192.

A proof of the existence of an orthogonal reference system moving along a periodic solution of $\dot{x} = X(x)$ so that one of the vectors remains tangent to the path; the system may be chosen so that the normal vectors move with the same degree of smoothness as X has; cf. S. P. Diliberto and G. Hufford [Contributions to the theory of nonlinear oscillations, vol. 3, pp. 207-236, Princeton Univ. Press, 1956; MR 18, 653]. A second constructive method is explained which is better adapted for computational purposes. Applications to stability, perturbations, etc. are considered.

J. L. Massera (Montevideo)

5921:

Urabe, Minoru. Geometric study of nonlinear autonomous oscillations. Funkcial. Ekvac. 1 (1958), 1-84.

In chapter I the results of the paper reviewed above are reproduced. Chapters II and III contain essentially known results on perturbations, stability and existence of perturbed periodic solutions by the method of the moving reference system. In chapter IV methods similar to Krylov-Bogolyubov's [Introduction to non-linear mechanics, Princeton Univ. Press, 1949; MR 4, 142] are used to find periodic solutions of systems $\dot{x} = X(x, \epsilon)$ when $\dot{x} = X(x, 0)$ has a closed path in the neighborhood of which all paths are closed. In chapter V a similar problem is discussed in which the unperturbed system has an m -parameter family of closed paths, $0 < m < n-1$. The case $n=2$ is discussed in great detail in chapters VI and VII; results on rotated vector fields such as Duff's [Ann. of Math. (2) 57 (1953), 15-31; MR 14, 751] appear as special cases.

J. L. Massera (Montevideo)

5922:

Yachter, Morris. On the existence of periodic solutions of the differential equation: $\dot{x} + F(x) = \epsilon f(x, \dot{x}, t; \epsilon)$. New York University, New York, 1956. 16 pp.

The variables x, t in the differential equation of the title are transformed into complex variables z , so that the differential equation assumes the form

$$\frac{d^2z}{d\tau^2} + az + b = \epsilon \varphi\left(z, \frac{dz}{d\tau}, \tau\right),$$

where φ is, in general, a multiple valued function of z . Conditions of periodicity of the solution $x(t)$ are expressed in terms of contour integrals on certain paths of a Riemann surface in the τ domain. These conditions for periodicity also yield a first approximation to the periodic

solution. Most of the paper is concerned with the case $F(x)=x-x^3$ and the results are too complicated to state here. The author claims that this procedure is easier to apply than the method of Poincaré. Since the paper is an abridgement of the author's thesis, it is very difficult to read, and, of course, no proofs are given in detail.

J. K. Hale (Baltimore, Md.)

5923:

Papoulis, A. Strongly non-linear oscillations. *J. Math. Phys.* 37 (1958), 147-156.

The differential equation is

$$(*) \quad \ddot{x} + w(x) + \varepsilon f(x, \dot{x}, t) = 0.$$

The reduced equation $\ddot{x} + w(x) = 0$ is assumed to have a known solution $r(t)$, and $t_1(t)$ is a function with the property that $x(t) = r(t_1(t))$ is a solution of (*). A series for $t_1(t)$ is given in terms of powers of ε . In the conservative case $f(x, \dot{x}, t) = f(x)$, the first order approximations for $t_1(t)$, $x(t)$, and the period of $x(t)$ are given explicitly. The forced oscillations when $f(x, \dot{x}, t) = -\sin \omega t$ and when $f(x, \dot{x}, t) = f(x, \dot{x}) - \sin(\omega t + \theta)$ are discussed. In the nearly linear case $w(x) + \varepsilon f(x, \dot{x}, t) = \omega_1^2 x + \varepsilon f(x)$ results of Krylov and Bogoliubov are obtained. The presentation is formal but under suitable conditions most of the statements made can probably be justified, and for some purposes the method, which is a modification of Rauscher's method, does have advantages.

J. P. LaSalle (Baltimore, Md.)

5924:

Proskuriakov, A. P. Investigation of the stability of the solution of a linear differential equation of the second order with periodic coefficients. *J. Appl. Math. Mech.* 22 (1958), 338-343 (250-253 *Prikl. Mat. Meh.*).

Using a method, which is essentially that of Hill, for the estimation of the characteristic exponents of a second-order linear differential equation with periodic coefficients, the author studies the stability of a system arising in the study of the motion of a propeller blade.

J. P. LaSalle (Baltimore, Md.)

5925:

Halanaĭ, A. [Halanay, A.] Some qualitative questions in the theory of differential equations with retarded arguments. *Rev. Math. Pures Appl.* 2 (1957), 127-144. (Russian)

Consider

$$(1) \quad \dot{x} = f(t, x(t), x(t-\tau)), \quad \tau > 0,$$

x, f vectors, and let $x(t; t_0, \varphi)$ be the solution defined for $t \geq t_0 - \tau$ by the initial condition $x(t) = \varphi(t)$, $t_0 - \tau \leq t < t_0$, φ a given continuous function in $[t_0 - \tau, t_0]$. § 2-5 of this paper are devoted to questions of existence of solutions and their dependence on the initial conditions; § 7 to the investigation of uniform stability and uniform asymptotic stability in the spirit of Lyapunov's second method; the latter results are very similar to those of N. N. Krasovskii [*Prikl. Mat. Meh.* 20 (1956), 315-327; *MR* 18, 128].

Other theorems are: 5. If (1) is linear, i.e., $f = A(t)x(t) + B(t)x(t-\tau)$, uniform asymptotic stability implies exponential stability ($\|x(t)\| \leq k \cdot e^{-\alpha(t-t_0)} \sup \|\varphi\|$, with fixed k, α). 6. If (1) is quasi-linear periodic,

$$f = A(t)x(t) + B(t)x(t-\tau) + \mu f(t, x(t), x(t-\tau)),$$

A, B, f periodic in t with period ω , μ small, and if $x=0$ is a uniform asymptotically stable solution of the system for $\mu=0$, a periodic solution exists for sufficiently small μ , its period being the least integral multiple $k\omega > \tau$. 7. If f depends on μ and has period ω in t and if (1) has a

periodic solution for $\mu=0$ such that the corresponding variational equations have no periodic solutions, there are periodic solutions for sufficiently small μ . 10. If f is periodic in t , any bounded and uniformly stable solution of (1) is asymptotically almost-periodic.

J. L. Massera (Montevideo)

5926:

Cooke, K. L. A symbolic method for finding integrals of linear difference and differential-difference equations. *Math. Mag.* 31 (1957/58), 121-126.

By symbolic methods, the author obtains a particular solution of the differential-difference equation

$$\sum_{j=0}^n \sum_{i=0}^m a_{ji} u^{(i)}(x+b_i) = f(x),$$

where $f(x)$ is a given function (an exponential, a sinusoidal, a polynomial, or a combination of these functions), a_{ji} real constants, and $0 = b_0 < b_1 < \dots < b_m$.

C. J. Bouwkamp (Eindhoven)

5927:

Zautykov, O. A. On a linear partial differential equation of first order with countably many independent variables and a countable number of parameters. *Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh.* 1 (1958), 25-40. (Russian)

Consider the equation $dx/dt = \omega(t, x, \lambda)$, where x, ω, λ are vectors with countably many components, as well as the equation $(\partial z / \partial t) + \sum a_k(t, x, \lambda)(\partial z / \partial x_k) = z f(t, x, \lambda) + g(t, x, \lambda)$, where z is a scalar and x_k are the components of x . Questions of existence and dependence on the initial conditions and parameters are investigated. The statements are extremely long and complicated and the whole approach is classical, not functional-analytical.

J. L. Massera (Montevideo)

PARTIAL DIFFERENTIAL EQUATIONS

See also 5927, 5976.

5928:

*Tricomi, Francesco G. *Equazioni a derivate parziali*. Edizioni Cremonese, Rome 1958. xii+392 pp. 5500 Lire.

In 1954, the author published his "Lezioni sulle equazioni a derivati parziali" [Editrice Gheroni, Torino; *MR* 16, 703], a valuable book marred by poor reproduction from typescript. The present work, now properly printed, is based on the author's lecture-course at Turin but is not a mere reproduction of the *Lezioni*. It is a completely rewritten book.

Whilst it does cover much the same ground as the *Lezioni*, there are certain important changes. The first chapter of the *Lezioni*, dealing with integral equations and special functions, has been deleted. This is little loss; no one would go to a book on partial differential equations for information on special functions; and no use is now made of the theory of integral equations, because the fundamental existence theorem for harmonic functions is now proved by Perron's method of subharmonic functions.

The first chapter (pages 1-83) deals with equations of the first order and the theory of characteristics. The Cauchy-Kowalevski theorem is not discussed, because the author does not regard it as providing an effectively useful basis for the general theory of partial differential equations.

Chapter II (pages 84-182) is concerned with equations of hyperbolic type of order 2. It is noteworthy that the difficult proof, given in the Lezioni, of the existence and uniqueness theorem for the problem of Goursat is greatly simplified by discussing the same problem for a hyperbolic system. The number of independent variables is not restricted to two when the general case involves no greater difficulty.

Chapter III (pages 183-319) deals with equations of elliptic type, in the main with the potential equation, though there is a short section on biharmonic functions.

Chapter IV (pages 320-382) discusses equations of parabolic or of mixed type, mainly the equation of heat conduction and Tricomi's equation of mixed type.

The whole book is a thoroughly up-to-date work on the classical approach to the theory of partial differential equations.

E. T. Copson (St. Andrews)

5929:

*Smirnow, W. I. *Lehrgang der höheren Mathematik*. IV. Hochschulbücher für Mathematik, Bd. 5. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. xii+708 pp. DM 40.00.

For a review of vol. 4 of the first Russian edition [OGIZ, Leningrad-Moscow, 1941] see MR 6, 42. The present volume is translated by C. Berg and L. Berg from the third Russian edition [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1953], which is not much different from the second [1951; MR 14, 145].

5930:

Kozlov, E. M. *Method for successive diminution of the order of a system of linear differential equations with slowly changing coefficients*. *Dopovidi Akad. Nauk Ukraïn. RSR* 1958, 813-816. (Ukrainian. Russian and English summaries)

"This note presents a method for integrating a system of linear differential equations with slowly changing coefficients, based on a successive diminution of the order of the system by means of partial solutions of certain auxiliary systems of differential equations. The partial solutions change slowly and are stable, thanks to which their construction may be carried out by a numerical integration with a comparatively large step."

Author's summary

5931:

Babuška, Ivo. *Über Schwarzsche Algorithmen in partiellen Differentialgleichungen der mathematischen Physik*. *Czechoslovak Math. J.* 8 (83) (1958), 328-343. (Russian. German summary)

This paper is concerned with the convergence question of the Schwarz algorithm for partial differential equations. For problems for union or intersection of two regions, and for general positive definite self-adjoint problems, the convergence proof of the Schwarz alternating algorithm is reduced to the following general result. Let H_1, H_2 be two closed linear subspaces of a Hilbert space H . Let P_1, P_2 denote the orthogonal projections of H onto H_1, H_2 , respectively, and let P be the projection onto $H_1 \cap H_2$. Then

$$\lim_{n \rightarrow \infty} \|(P_2 P_1)^n x - Px\| = 0$$

for all $x \in H$. (This general theorem was first proved by J. von Neumann [*Functional operators. II. The geometry of orthogonal spaces*, Princeton Univ. Press, 1950; MR 11, 599; p. 55] and later independently by N. Wiener [Comment. Math. Helv. 29 (1955), 97-111; MR 16, 921].)

Ky Fan (Notre Dame, Ind.)

5932:

*Titchmarsh, E. C. *Eigenfunction problems arising from differential equations*. Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 393-403. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. \$7.00.

This paper presents a compact survey of the extensive recent work of the author on the theory of ordinary and partial differential equations and associated expansion problems, and then as specific examples of the general theory considers the quantum mechanics problems of the hydrogen atom, the helium atom, and the hydrogen atom in a weak electric field.

W. T. Reid (Evanston, Ill.)

5933:

Il'in, V. A. *Sufficient conditions for expansibility of a function in an absolutely and uniformly convergent series of eigenfunctions*. *Mat. Sb. N. S.* 46(88) (1958), 3-26. (Russian)

The eigen-functions in question are those of $\Delta u + \lambda u = 0$ in an N -dimensional region, with any of the standard boundary conditions. The first topic is the absolute and uniform convergence of the Fourier expansion of an $f(Q)$ with a singularity at an internal point P of the form $f(Q) = r_{PQ}^\epsilon + v(Q)$ ($\epsilon > 0$), or $f(Q) = r_{PQ}^{2m} \cdot \ln r_{PQ} + v(Q)$ ($m = 1, 2, \dots$), where $v(Q)$ is suitably smooth; that absolute convergence fails when $\epsilon < 0$ or $m = 0$ is illustrated by considering, in more detail than previously [Dokl. Akad. Nauk SSSR 74 (1950), 413-416; MR 13, 350], a Green's function for a rectangle. The author next gives a full account of absolute and uniform convergence of the Fourier expansion of a function represented source-wise by an integral over a k -dimensional manifold of Lyapunov type; the result was stated without proof in Dokl. Akad. Nauk SSSR 105 (1955), 210-213 [MR 17, 744], where only the case $N=2$ was proved. For the latter case, the author expands his previous remarks on the expansibility of functions with discontinuous derivatives.

F. V. Atkinson (Canberra City)

5934:

Laurenti, Fernando. *Considerazioni geometriche sopra una notevole equazione differenziale*. *Archimede* 10 (1958), 126-129.

Expository treatment of the equation $u \times dP = 0$, where the vector u is a given function of the variable point P .

5935:

Halmovič, Mendel'. *Quelques propriétés des éléments intégraux d'un système de Pfaff du IIe genre*. *Ž. Čist. Prikl. Mat.* 1 (1956), 23-33. (Russian)

Translated from the Romanian [Acad. R. P. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz. 7 (1955), 301-311; MR 17, 740; a French version is given in Rev. Math. Pures Appl. 1 (1956), no. 1, 23-32; MR 19, 147].

5936:

Castoldi, Luigi. *Attorno a un teorema Pfaffiano di Carathéodory*. *Rend. Sem. Fac. Sci. Univ. Cagliari* 27 (1957), 204-209.

The author considers a differential form of Pfaff, namely,

$$D(v, dP) = \sum_{i=1}^n v_i dx_i,$$

where the v_i are regular vector functions of $x = (x_1, \dots, x_n)$

in a certain region X_n of n -dimensional space. It is supposed that in every arbitrarily small neighborhood of a point P_0 of X_n there is a point P for which there is no curve C passing through P_0 and P which is an integral curve of the Pfaffian equation $D(v, dP)=0$. Such points P are called inaccessible points from P_0 . Carathéodory showed that under these hypotheses the Pfaffian equation is integrable. The author gives the following alternative result. The Pfaffian equation $D(v, dP)=0$ is integrable or not, according as the totality of points P accessible from an arbitrary point P_0 of X_n generate an X_{n-1} passing through P_0 or else the interior of X_n . Applications are made to thermodynamics and mechanics.

J. De Cicco (Chicago, Ill.)

5937:

Mirer, V. S. The Cauchy problem for simultaneous linear partial differential equations with analytic coefficients. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 873-875. (Russian)

The present note considers the question of convergence of the simplest difference scheme for approximating the system of equations

$$\frac{\partial U_\alpha}{\partial t} = \sum_{\beta=1}^N F_{\alpha\beta}(t, x) \frac{\partial U_\beta}{\partial x} + B_\alpha(t, x) \quad (\alpha=1, \dots, N),$$

where the functions $F_{\alpha\beta}, B_\alpha$ are analytic on a closed region G of the (t, x) plane. The initial conditions

$$U_\alpha(t_0, x) = \phi_\alpha(x) \quad (\alpha=1, \dots, N)$$

are given on an interval $G_0 \subset G$ of the straight line $t=t_0$, the $\phi_\alpha(x)$ being analytic on this interval.

From the introduction

5938:

Lebedev, V. I. On the mesh method for a certain system of partial differential equations. Izv. Akad. Nauk SSSR. Ser. Mat. 22 (1958), 717-734. (Russian)

The system is a slight generalization of one discussed by Sobolev [same Izv. 18 (1954), 3-50; MR 16, 1029]. The properties of the solution are developed through a study of the solution of the difference analogue.

A. S. Householder (Oak Ridge, Tenn.)

5939:

Saltikov, N. Note sur la théorie des caractéristiques d'équations aux dérivées partielles du premier ordre. Glas Srpske Akad. Nauka. 232 Od. Prirod.-Mat. Nauka (N.S.) 15 (1958), 9-19. (Serbo-Croatian. French summary)

L'auteur analyse l'algorithme Jacobien pour former les équations des caractéristiques d'équations aux dérivées partielles du premier ordre à une fonction inconnue. Il y introduit deux nouvelles notions, de corrélation et d'incompatibilité d'équations aux dérivées partielles avec leurs intégrales complètes.

B. S. Popov (Skopje)

5940:

Breuer, Manfred. Jacobische Differentialgeometrie und Systeme partieller Differentialgleichungen 1. Ordnung. Dissertation. Bonn. Math. Schr., no. 7 (1958), 74 pp.

The author calls a Jacobi manifold a pair consisting of a manifold \mathcal{V} of class C^r , and a twice contravariant, non-singular, skew symmetric tensor A , and defines a Jacobi curvature tensor J , essentially the exterior derivative of the 2-form determined by the inverse A^* of A . [These concepts were introduced earlier by H. C. Lee, Amer. J. of Math. 65 (1943), 433-438; 67 (1945), 321-328; MR 5, 15; 7, 81; although without discussion of differentiability conditions.] The main purpose of the author is to give a

geometric interpretation and generalization of the theory of systems of first order partial differential equations. After the study of some motivating cases this takes the following form: A submanifold \mathcal{W} of the Jacobi manifold (\mathcal{V}, A) , with inclusion map i , is called A -complete if $A(\text{kernel } i^*) \subset \text{Image } i_*$; here i_* , i^* are the induced maps on tangent covectors, resp. vectors, and A is considered as a map from vectors to covectors at any point of \mathcal{V} . Similarly, \mathcal{W} is called a A^* -strip (=Streifen) manifold if $A^*(\text{Image } i_*) \subset \text{Kernel } i^*$. A first order partial differential system becomes the problem: given an A -complete submanifold \mathcal{W} of (\mathcal{V}, A) , to find A^* -strip manifolds \mathcal{L} , contained in \mathcal{W} . For the initial value problem suppose given a strip manifold \mathcal{ACW} with $\dim \mathcal{W} - \dim \mathcal{A} = \frac{1}{2} \dim \mathcal{V}$; to find a strip manifold \mathcal{L} with \mathcal{ACLW} . The solution depends on the concept of characteristic field $\Gamma_{\mathcal{W}}$: this is the contact field, or distribution, of dimension m in \mathcal{W} determined by $i_*(\Gamma_{\mathcal{W}}(p)) = A(\text{Kernel } i_p^*)$ at any point p . The field $\Gamma_{\mathcal{W}}$ is completely integrable. The solution of the initial value problem consists, roughly speaking, in passing through each point of \mathcal{A} the integral manifold of $\Gamma_{\mathcal{W}}$, assuming that \mathcal{A} is transversal to $\Gamma_{\mathcal{W}}$. Since the author's main interest is in the global aspect of the theory, this requires a detailed investigation of foliated manifolds, and the corresponding quotient "manifolds". In fact, a large part of the paper consists of a detailed study of the concepts of differentiable manifold, vector field, differentiable map, differentiable foliations, etc. The connections with canonical transformations, canonical coordinate systems, Poisson- and Lagrange brackets, Jacobi Hamilton theory, symplectic manifolds are treated.

H. Samelson (Ann Arbor, Mich.)

5941:

Parsons, D. H. The extension of Monge's method to singular systems. J. Math. Pures Appl. (9) 37 (1958), 135-152.

Consider the system

$$(1) \quad f_i(x, y, z_j, p_j, q_j) = 0, \quad p_j = \frac{\partial z_j}{\partial x}, \quad q_j = \frac{\partial z_j}{\partial y}, \quad i, j = 1, \dots, n,$$

where the f_i are continuous functions of their arguments, with continuous partial derivatives, in a neighbourhood of a set of values satisfying the system, and where the equations are of fully reduced form; suppose that the system cannot by any change of variables be reduced to a system with less than n dependent variables. Let M be the $n \times n$ matrix (d_{ij}) , where $d_{ij} = (\partial f_i / \partial p_j) dy - (\partial f_i / \partial q_j) dx$, and let $\Delta = |M|$. If $\Delta \neq 0$, one can eliminate the p_j and q_j between (1) and (2) $dz_j - p_j dx - q_j dy = 0$, $j = 1, \dots, n$, so that if m is chosen to be any function of x, y , and z_j in (3) $dy - m dx = 0$, one obtains a system of characteristics in the usual sense. The author shows that certain choices of m lead to useful results for the singular system (1) with $\Delta = 0$. If, in general, $\text{rank } (M) = n - r$, $r \geq 1$, one obtains a system of total differential equations in x, y, z_j . If one can obtain at least one more such equation, (3) is said to define a system of characteristics of order 0; if (3) implies $\text{rank } (M) = n - s$, $s \geq r$, this system is of rank s . Theorem: If the system of characteristics is of order 0 and rank s , with an integrable combination $du = 0$, then $r + s \leq n$, and (1) may be put in the form $f_i(x, y, z_j, p_j) = 0$ ($i = 1, \dots, s$), $g_i(x, y, z_j, q_j) = 0$ ($i = 1, \dots, r$), $h_i(x, y, z_j, p_j, q_j) = 0$ ($i = 1, \dots, n - r - s$). More generally, a canonical form for (1) is given when the system of characteristics admits $t+1$ integrable combinations, where $t \leq s$. An example is given.

W. J. Cales (Salt Lake City, Utah)

5942:

Parsons, D. H. Linear singular systems of three partial differential equations. *J. Math. Pures Appl.* (9) 37 (1958), 265-268.

This review is in the language of the preceding review. Let (1) be linear with $\Delta=0$ and $n=3$. It is shown that $r=1$, and that, on elimination of the p 's and q 's, the total differential form obtained is either a form of Pfaff or a quadratic form in the differentials, of rank 4. Hence, by previous work of the author, there is a general existence theorem for this system, and its integration can be reduced to the integration of an auxiliary non-singular system. *W. J. Coles* (Salt Lake City, Utah)

5943:

Lopatinskiĭ, Ya. B. Uniqueness of the solution of Cauchy's problem for an equation of the Schrödinger type. *Dopovidi Akad. Nauk Ukraïn. RSR* 1958, 119-122. (Ukrainian. Russian and English summaries)

"The author proves the uniqueness of a solution of Cauchy's problem for the equation $\Delta u + f(x)u = 0$ [$x = (x_1, x_2, x_3)$], making the assumption that $f(x)$ is a continuous complex-valued function." *Author's summary*

5944:

Kravtchenko, Julien; et Apté, Achyut. Note sur la méthode d'intégration de Fourier des équations de la physique mathématique. *Ann. Inst. Fourier. Grenoble* 7 (1957), 329-358.

La méthode de Fourier, qui consiste à chercher le développement des fonctions cherchées en série de fonctions propres, est souvent employée dans la solution de différents problèmes aux limites sans justification théorique. Cette justification, connue depuis longtemps, tout au moins dans les cas les plus intéressants, met en oeuvre souvent des moyens d'un niveau assez élevé pour rebuter les non-spécialistes. Le but du mémoire des auteurs est de répandre ces démonstrations par des considérations assez élémentaires. Ils donnent en exemple la détermination d'une solution de l'équation $\Delta F + k^2 F = 0$, régulière dans le rectangle D défini par $0 \leq x \leq a$, $0 \leq y \leq b$, et sur C , la frontière de D , assujettie à vérifier les conditions aux limites $\partial F / \partial r = 0$, partout sur C à l'exception du segment $y=0$, $\alpha < x < \beta$ où $\partial F / \partial r = f(x)$. Admettant qu'on ait pu étudier a priori les propriétés de F et s'assurer que cette fonction admet un développement en série de fonctions propres, on peut par un procédé approprié calculer les coefficients de ce développement et examiner la convergence de la série obtenue. Mais alors surgit une autre difficulté. Pour vérifier que la fonction satisfait aux conditions aux limites il faut calculer les dérivées $\partial F / \partial x$ et $\partial F / \partial y$, ce qu'on ne saura faire pour les points de C en dérivant le développement trouvé terme à terme. Pour le problème qu'ils donnent en exemple, ils montrent comment surmonter ces difficultés. Ils prennent $f(x) = \varphi(x)(x-\alpha)^{-\nu}(\beta-x)^{-\nu}$, $\varphi(x)$ étant une fonction analytique, bornée pour $\alpha \leq x \leq \beta$ et $0 \leq \nu < \frac{1}{2}$ (ces conditions étant choisies plus restrictives que la solution du problème ne l'exige, pour faciliter les démonstrations). Pour quelques détails des plus délicats ils sont obligés de renvoyer à la littérature citée à la fin du mémoire.

H. Bremekamp (Delft)

5945:

Martin, A. I. Some further work on L^2 -solutions of the wave equation. *Quart. J. Math. Oxford Ser. (2)* 7 (1956), 280-286.

The present paper is a continuation of earlier work of

the author [same *J.* 5 (1954), 212-227; *MR* 16, 827] on the existence of L^2 -solutions of the differential equation

$$(*) \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + (\lambda - q(x, y))\phi = 0,$$

where $\phi(x, y)$ is a real-valued continuous function, and the region of consideration is the entire (x, y) -plane. As in the earlier paper, basic to the author's treatment are the results of Titchmarsh [*Proc. London Math. Soc.* (3) 1 (1951), 1-27; *MR* 13, 241] on the existence and properties of a Green's function $G(x, y, \xi, \eta, \lambda)$ for (*). If $G(x, y, \xi, \eta, \lambda)$ is a Green's function for (*), and

$$H(x, y, \xi, \eta, u) = (1/\pi) \lim_{u \rightarrow 0} \int_0^u \operatorname{Im} G(x, y, \xi, \eta, u' + iv) du',$$

then the following specific results are established: (1) if μ is a real value at which some Green's function is such that the corresponding H -function is discontinuous, then as a function of (x, y) the saltus function

$$\phi(x, y, \xi, \eta) = H(x, y, \xi, \eta, \mu + 0) - H(x, y, \xi, \eta, \mu - 0)$$

is an L^2 -solution of (*) for $\lambda = \mu$; (2) if $G(x, y, \xi, \eta, \lambda)$ is not unique for some (and hence all) non-real λ , and μ is a real value that is interior to an open interval throughout which some corresponding $H = H(u)$ is constant, then (*) admits a non-trivial L^2 -solution for $\lambda = \mu$. Finally, for the particular case of $q(x, y)$ a function $q(r)$ of the radius vector $r = (x^2 + y^2)^{1/2}$ such that the Green's function for (*) is not unique, it is shown that (*) has a non-trivial L^2 -solution for every real λ . *W. T. Reid* (Evanston, Ill.)

5946:

Bergman, Stefan. Operators generating solutions of certain partial differential equations in three variables and their properties. I. *Scripta Math.* 23 (1957), 143-151 (1958).

Let Δ denote the three-dimensional Laplace operator, and let $F = F(y, z)$ be independent of x . The author uses his well-known integral operator method to generate two different classes of solutions of the partial differential equation $\Delta u + Fu = 0$, and to characterize the singularities of the solutions thus obtained.

Z. Nehari (Pittsburgh, Pa.)

5947:

Bureau, F. J. Divergent integrals and partial differential equations. *Advancement in Math.* 3 (1957), 271-324. (Chinese)

A translation from the English in *Comm. Pure Appl. Math.* 8 (1955), 143-202 [*MR* 16, 826].

5948:

Malaviya, S. C. Solution of Cauchy's problem for the wave equation $(\partial^2 / \partial t^2 - \nabla^2 + k^2)\psi = 0$. *Proc. Indian Acad. Sci. Sect. A.* 48 (1958), 190-196.

The author extends Copson's method of solving the wave equation for any odd number of spatial dimensions [*Proc. Roy. Soc. London Ser. A* 235 (1956), 560-572; *MR* 18, 46] to the equation of damped waves.

E. T. Copson (St. Andrews)

5949:

Tong, Kwang-chang. On a boundary value problem for the wave equation. *Sci. Record (N.S.)* 1 (1957), 277-278.

Certain boundary value problems for the wave equation $u_{xx} + u_{yy} - u_{zz} = 0$ were solved by the reviewer [Contributions to the theory of partial differential equations, pp. 249-257, Princeton Univ. Press, 1954; *MR* 16, 711]. Let D be the domain bounded by the three surfaces

$x^2 + y^2 = (x - z_0)^2 + (y - y_0)^2 = z^2, z = 0$. We denote by S_1 the portion of $(x - z_0)^2 + (y - y_0)^2 = z^2$ coinciding with D , by S_2 the part of $x^2 + y^2 = (z - z_0)^2$, and by S_3 the circle $x^2 + y^2 \leq z_0^2, z = 0$. In the paper quoted the boundary value problem with the function u prescribed on S_2 and S_3 was solved. It was stated (without proof) that the problem with data given on S_1 and S_3 could be solved by essentially the same method. The author shows by a remarkably simple explicit example that this last statement is incorrect. A solution of the wave equation in D is exhibited which vanishes on S_1 and S_3 and which does not vanish identically. This fact could also be deduced from the result of F. John, "Plane waves and spherical means applied to partial differential equations", Interscience Publishers, New York-London, 1955 [MR 17, 746], p. 119.

M. H. Protter (Berkeley, Calif.)

5950:

Wang, Guang-ying. The Goursat problems in space. Sci. Record (N.S.) 1 (1957), 283-286.

The author establishes a uniqueness theorem for the Goursat problem described in the preceding review with values given on S_2 and S_3 . In addition, the problem with the normal derivative prescribed on S_3 in place of the unknown function is also solved. One considers the expression $v \square u - u \square v$, where \square is the wave operator in n dimensions, and then solves an appropriate associated problem for the function v . The author remarks that in the proof of this result given by the reviewer [J. Rational Mech. Anal. 3 (1954), 435-446; MR 16, 43] the integral I_3 occurring in the proof does not vanish as asserted. The observation is correct, but the integral in question is dominated by I_4 (p. 440), and hence the proof referred to remains valid.

M. H. Protter (Berkeley, Calif.)

5951:

Copson, E. T.; and Erdélyi, A. On a partial differential equation with two singular lines. Arch. Rational Mech. Anal. 2 (1958), 76-86.

In an earlier paper Copson [same Arch. 1 (1958), 349-356] investigated the partial differential equation

$$(*) \quad \frac{\partial^2 U}{\partial x^2} + 2\alpha \frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial y^2} + \frac{2\beta}{y} \frac{\partial U}{\partial y},$$

which has two singular lines. Subject to the conditions that the solutions are continuously differentiable in the closed quadrant $x \geq 0, y \geq 0$ and twice differentiable in the open quadrant $x > 0, y > 0$, it turns out that they must satisfy the supplementary conditions

$$\frac{\partial U}{\partial x} = 0, x = 0, y \geq 0 \quad \text{and} \quad \frac{\partial U}{\partial y} = 0, y = 0, x \geq 0.$$

If one attempts to assign the function U on either axis, one finds a functional relation between them.

Instead of using the method of Riemann to solve (*) as Copson did in the earlier paper, the authors now use the methods of fractional integration and the Mellin transformation. Each of these methods enables the authors to recover the functional relation referred to and to obtain some other results which complement the earlier work.

A. E. Heins (Pittsburgh, Pa.)

5952:

Povzner, A. Ya.; and Suharevskii, I. V. On the discontinuities of Green's function in a mixed problem for the wave equation and some diffraction problems. Dokl. Akad. Nauk SSSR 122 (1958), 986-989. (Russian)

Let D be a simply connected (possibly unbounded) domain in the plane bounded by an infinitely differen-

tiatile contour S . The solution of the wave equation $\Delta u = u_{tt}$ with boundary conditions $u(0, x) = 0, u_t(0, x) = f(x), u|_S = 0$ can be given in the form

$$u(t, x) = u_0(t, x) + \int_D w(t, x, y) f(y) dy.$$

Here $u_0(t, x)$ is the solution of the Cauchy problem for the entire plane, with $f(x)$ defined as zero outside D . The authors are concerned with the general behavior and discontinuities of $w(t, x, y)$. Using the intimate connection between w and the Green's function for the Dirichlet problem for $\Delta v + kv = 0$, they set up an integral equation and a procedure for obtaining successive approximations to w .

A. N. Milgram (Minneapolis, Minn.)

5953:

Zaidman, Samuel. Sur la presque-périodicité des solutions de l'équation non homogène des ondes. C. R. Acad. Sci. Paris 247 (1958), 2276-2278.

The author proves for the first time for the inhomogeneous case the following theorem which for the homogeneous case had been previously treated several times and even in settings more general than his [C. F. Muckenhoupt, J. Math. Phys. 8 (1929), 163-198; S. Bochner, Acta Math. 62 (1934), 227-237; S. Bochner and J. v. Neumann, Ann. of Math. (2) 36 (1935), 255-290; see also S. Soboleff, Dokl. Akad. Nauk SSSR 48 (1945), 542-545, 618-620; 49 (1945), 12-15; MR 8, 78]. Consider the equation

$$u_{tt}(X, t) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(X) \frac{\partial u}{\partial x_j} \right) - a(X) u(X, t) + f(X, t),$$

with $a_{ij} > 0$ and $a(X) > 0$ in a regular domain in R^n with boundary condition $u|_S = 0$, and for t in $-\infty < t < \infty$, and assume that, with regard to X , the inhomogeneous datum $f(X, t)$ and the solution $u(X, t)$ have values in the Banach space L_2 . Now, if the L_2 -ranges of $u(X, t)$ and $u_t(X, t)$ have compact closure and if $f(X, t)$ is almost periodic in t , then $u(X, t)$ is likewise almost periodic in t .

S. Bochner (Princeton, N.J.)

5954:

Kordunyanu, K. [Corduneanu, C.] La dépendance des solutions des équations hyperboliques par rapport aux coefficients et aux données sur les caractéristiques. Z. Čist. Prikl. Mat. 1 (1956), 45-49. (Russian)

A translation from the Romanian in Acad. R. P. Romine. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 313-317 [MR 17, 1214]; a French translation was published in Rev. Math. Pures Appl. 1 (1956), no. 1, 41-44 [MR 19, 149].

5955:

Vylkovič, V. [Vălcovici, V.] Sur le théorème des valeurs extrêmes. Z. Čist. Prikl. Mat. 1 (1956), 35-43. (Russian)

A translation from the Romanian in Acad. R. P. Romine. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 741-749 [MR 17, 491]; a French translation was published in Rev. Math. Pures Appl. 1 (1956), no. 1, 33-40 [MR 19, 150].

5956:

Pucci, Carlo; e Weinstein, Alessandro. Sull'equazione del calore con dati subarmonici e sue generalizzazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 493-496.

The authors derive some monotonicity properties of solutions of the heat equation in a strip, having subharmonic initial values.

A. Friedman (Berkeley, Calif.)

5957:

Geeslin, Roger H. Note on a class of diffusion equations. Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 788-791.

The paper is an abstract of the author's dissertation which discusses the Cauchy problem for the diffusion equation (1): $(b(x)T_x)_x - T_t = 0$ in the space $L(-\infty, \infty)$, $b(x)$ being positive and continuous. Suggested by a paper of E. Hille [J. Analyse Math. 3 (1954), 81-196; MR 16, 45], two cases are considered: Case 1, $B(-\infty) < \infty$ and $B(\infty) < \infty$; and case 2, $B(-\infty) = B(\infty) = \infty$; where $B(x)$ denotes $\int_0^x sb(s)^{-1}ds$. Let $\{w_n(x)\}$ and $\{\lambda_n\}$ be the characteristic functions and characteristic values of the equation $Lu(x) = (b(x)u'(x))' = \lambda u(x)$. Then, in case 1,

$$(2): T(t)f(x) = \int_{-\infty}^{\infty} K(x, s, t)f(s)ds,$$

$$K(x, s, t) = \sum_{n=1}^{\infty} w_n(x)w_n(s)\exp(\lambda_n t),$$

gives the solution of the Cauchy problem for every $f \in D(L)$. Under the further restrictions: $\int_{-\infty}^{\infty} |w_n(x)|dx < C_1|\lambda_n|^p$ and $\sup_x |w_n(x)| < C_2|\lambda_n|^p$ ($p > 0$), (2) is the solution of the Cauchy problem for every $f \in L(-\infty, \infty)$. If we assume that L is the infinitesimal generator of a semi-group of positive contraction operators on $L(-\infty, \infty)$ — such is the case in the case 2 — and the resolvent of L has the representation $R(\lambda, L)g(x) = \int_{-\infty}^{\infty} G(x, s, \lambda)g(s)ds$ ($\text{Re}(\lambda) > 0$) with symmetric and continuous kernel G , then $G(x, s, \lambda)$ is completely monotonic and $K(x, s, t) = \lim_{\sigma \rightarrow \infty} (2\pi i)^{-1} \int_{\sigma-i\infty}^{\sigma+i\infty} G(x, s, \lambda)e^{t\lambda}d\lambda$ for $t > 0$ and for any fixed $c > 0$. If $\int_{-\infty}^{\infty} \lambda G(x, s, \lambda)ds \leq 1$ ($\text{Re}(\lambda) > 0$), then (2) gives the solution of the Cauchy problem for every $f \in L(-\infty, \infty)$. K. Yosida (Tokyo)

5958:

Itô, Seizô; and Yamabe, Hidehiko. A unique continuation theorem for solutions of a parabolic differential equation. J. Math. Soc. Japan 10 (1958), 314-321.

The authors prove a weak form of the unique continuation theorem for solutions of (1) $\partial u(t, x)/\partial t = Au(t, x)$ ($0 < t < \infty$, $x \in D$), where A is the elliptic operator

$$Au = \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} \left(\sqrt{a(x)} a^{ij} \frac{\partial}{\partial x^j} u \right) + c(x)u,$$

$-\infty < c(x) \leq C < \infty$, and the domain $D \subseteq R^m$ is not necessarily bounded. The boundary B of D is to consist of countably many C^3 -hypersurfaces of dimension $m-1$ and the solutions of (1) are to satisfy

$$(2) \quad \alpha(\xi)u + \{1 - \alpha(\xi)\}\partial u/\partial n_\xi = 0 \text{ for } \xi \in B,$$

where $0 \leq \alpha(\xi) \leq 1$. In addition to regularity assumptions on $\alpha(\xi)$ and the coefficients of A , it is assumed that (1) has a unique solution $u(t, x) \in L^2(D)$ satisfying (2) and $\lim_{t \rightarrow 0} \|u\|_2 = 0$ (where $\|u\|_2$ denotes the $L^2(D)$ norm of u). The latter assumption is assured, for example, when A is uniformly elliptic in D . The main theorem states that if $u(t, x)$ is a solution of (1) satisfying (2) which is in $L^2(D)$ for any $t > 0$, and if there exist $t_0 > 0$ and an open set $D_0 \subset D$ such that $u(t_0, x) = 0$ in D_0 , then $u(t, x) = 0$ for all $t > 0$ and $x \in \bar{D}$. The proof employs a fundamental solution constructed by Itô [Jap. J. Math. 27 (1957), 55-102; MR 20 #4702] and eigenfunction expansion theorems of Gårding [Tolft Skandinavisk Matematikerkongressen, Lund, pp. 44-55; MR 17, 158], Browder [Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 454-459, 459-463; 42 (1956), 769-771, 870-872; MR 16, 134; 19, 1061], and others. Previously, the same theorem had been proved by Yamabe [Ann. of Math. (2) 69 (1959), 462-466; MR 21 #206]. M. Schechter (New York, N.Y.)

5959:

Savin, G. M.; and Feščenko, S. F. On the asymptotic solution of a class of partial differential equations with variable boundary conditions. Dopovidi Akad. Nauk Ukraïn. RSR 1958, 588-594. (Ukrainian. Russian and English summaries)

The equation considered in this paper is

$$(1) \quad A \frac{\partial^2 u}{\partial t^2} = B \frac{\partial^2 u}{\partial x^2} + \varepsilon \left[C \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial t} + F \frac{\partial^2 u}{\partial x \partial t} + Ku + G e^{i\theta(t, \varepsilon)} \right]$$

with the initial and boundary conditions:

$$(2) \quad u = \phi(x), \quad \frac{\partial u}{\partial t} = \psi(x) \text{ for } t=0;$$

$$\varepsilon \left[a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial t} \right] + f_1 u = 0 \text{ for } x=0,$$

$$(3) \quad \varepsilon \left[a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial t} \right] + f_2 u = 0 \text{ for } x=1;$$

where $u = u(x, t)$; A, B, C, E, F, K and G are functions of τ, x and ε ; ε is a small positive parameter, $\tau = \varepsilon t$; a_1, a_2, b_1, b_2 , and f_1, f_2 are functions of τ and ε . The authors give asymptotic solutions of the problem subject to some additional conditions. S. Kulik (Logan, Utah)

5960:

Volpert, A. I. Investigation of boundary problems for elliptical systems of differential equations on a plane. Dokl. Akad. Nauk SSSR (N.S.) 114 (1957), 462-464. (Russian)

The author begins with the following boundary value problem: Let D be a simply connected domain in the plane with Hölder continuous boundary, Γ its boundary, $A_{kl}(z)$ real square matrices of order p , Hölder continuous for $k+l \leq n$, once continuously Hölder differentiable for $k+l = n$; $a_{kl}(t)$ and $b_{kl}(t, t_1)$ matrices of type $(\frac{1}{2}np \times p)$ with a_{kl} and $|t - t_1|^\alpha b_{kl}(t, t_1)$ Hölder continuous for t and t_1 on Γ . Consider the elliptic system

$$\sum_{k+l \leq n} A_{kl}(z) \frac{\partial^{k+l} U}{\partial x^k \partial y^l} = F(z), \quad z \text{ in } D,$$

with boundary conditions,

$$\sum_{k+l \leq n-1} [a_{kl}(t)U_{kl}(t) + \int_{\Gamma} b_{kl}(t, t_1)U_{kl}(t_1)dt_1] = f(t),$$

$$t \in \Gamma, U_{kl}(t) = D_{\nu}^{k+l} U.$$

By a substitution $u_k = U_{k-1, n-k}$, the writer obtains a formally equivalent system of the form

$$Lu = A(z) \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + B(z)u + \int_D R(x, \zeta)u(\zeta)dA_{\zeta} = \bar{F}(z) + M(z)c$$

with boundary system of the form

$$\Lambda u = a(t)u(t) + \int_{\Gamma} b(t, t_1)u(t_1)ds_1 = f(t) + N(t)c, \quad t \in \Gamma,$$

where u is a column vector of dimension $r = \frac{1}{2}np$. Using the ellipticity of the original system, he observes that $A(z)$ may be made skew-symmetric.

Relying upon methods previously applied by I. N. Vekua to second order equations [I. N. Vekua, "New methods for the solution of elliptic equations", OGIZ, Moscow 1948; MR 11, 598] and using the fundamental

solution for elliptic systems constructed by Ya. B. Lopatinski [Ukrain. Mat. Z. 3 (1951), 3-38, 290-316; 5 (1953), 123-151; MR 16, 256, 928; 17, 494] the author states a number of results concerning this new system.

Theorem 1 states that every solution of the homogeneous equation $Lu=0$ can be written uniquely in the form

$$u(z) = \int_{\Gamma} M(z, \zeta) \mu(\zeta) d\zeta + \sum_{1 \leq j \leq r} c_j u_j(z),$$

where $M(z, \zeta)$ is constructed in terms of the fundamental solution, μ is Hölder continuous on Γ , c_j are constants, and u_j are special solutions. Assume $\text{Det}(a(t)) \neq 0$ on Γ . Then theorem 2 states that the second boundary value problem has a solution if and only if the corresponding homogeneous problem has $m+r$ solutions which are linearly independent, where $m = \pi^{-1}[\arg \det(a(t))]_{\Gamma}$. If the adjoint system is definable, theorem 3 states that the difference between the linear dimension of the null solutions of the original problem and that of the adjoint problem is exactly $m+r$. Theorem 4 states that a necessary and sufficient condition for the solvability of $Lu=F$, $Lu=0$ is the orthogonality of F to the null solutions of the adjoint problem. The note concludes with the remark that under the hypotheses of theorem 2 it is possible to segregate a special family of solutions which vary continuously with the data.

F. Browder (New Haven, Conn.)

5961:

Topolyans'kii, D. B. On the upper and lower functions of Chaplygin in some boundary problems. *Dopovidi Akad. Nauk Ukrain. RSR* 1958, 361-363. (Ukrainian. Russian and English summaries)

It is shown that the set of lower functions of Chaplygin is a generalization of a class of subharmonic functions. Chaplygin's results are shown to apply to boundary problems with mixed conditions. The properties of a class of subharmonic functions and lower functions of Chaplygin are considered in connection with the bi-harmonic problems of Mathieu-Lauricella and Mathieu-Riquier.

H. P. Thielman (Ames, Iowa)

5962:

Miranda, Carlo. Sul teorema del massimo modulo per le equazioni lineari ellittiche in due variabili a coefficienti reali. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 24 (1958), 131-134.

For solutions of the second-order elliptic equation $\mathcal{L}(u)=f$ the inequality

$$(*) \quad \max_T |u| \leq \max_{\mathcal{F}T} |u| + k \max_T |f|$$

holds, under suitable hypotheses on the domain T and the coefficients of \mathcal{L} ; here \max_T and $\max_{\mathcal{F}T}$ denote the maxima over T and its boundary, respectively, while k denotes a constant depending only on T and the coefficients of \mathcal{L} . The author outlines briefly the proof of an analogous theorem for elliptic equations (in two independent variables) of order $2m$, $m > 1$, the term $\max_T |u|$ being replaced by a sum involving u and its derivatives up to the $(m-1)$ st order, while $\max_{\mathcal{F}T} |u|$ is replaced by a sum involving u and its normal derivatives of the first $(m-1)$ orders on the boundary. A detailed proof is to appear later.

Bernard Epstein (Philadelphia, Pa.)

5963:

Hua, Loo-keng. On a system of partial differential equations. *Sci. Record (N.S.)* 1 (1957), 369-371.

The author asserts as follows. In the space of n^2 com-

plex variables $Z=(z_{ij})$, $i, j=1, \dots, n$, if R denotes the domain $ZZ^* < 1$, if $\{U\}$ denotes the unitary matrices $UU^*=1$, and $d(U)$ the group invariant volume element on $\{U\}$ with total volume 1, then for any $\varphi(U)$ which is defined and continuous on $\{U\}$, the integral

$$u(Z) = \int_U \frac{\det(I-ZU^*)^n \varphi(U) d(U)}{|\det(I-ZU^*)|^{2n}}$$

satisfies the system

$$\sum_{\alpha, \beta=1}^n \left(\delta_{\alpha\beta} - \sum_{k=1}^n \bar{z}_{k\alpha} z_{k\beta} \right) \frac{\partial^2 u}{\partial \bar{z}_{i\alpha} \partial z_{j\beta}} = 0, \quad i, j=1, \dots, n$$

with boundary values $\varphi(U)$. The system was introduced by J. Mitchell [Trans. Amer. Math. Soc. 79 (1955), 401-422; MR 17, 253].

S. Bochner (Princeton, N.J.)

5964:

Zagors'kii, T. Ya. Certain boundary problems for a system of differential equations of the parabolic type with changing coefficients. *Dopovidi Akad. Nauk Ukrain. RSR* 1958, 364-367. (Ukrainian. Russian and English summaries)

The author gives a solution of a parabolic system of partial differential equations with variable, but sufficiently smooth, coefficients depending on the position coordinates as well as on time. The solution is required to satisfy certain boundary conditions.

H. P. Thielman (Ames, Iowa)

5965:

Borovikov, V. A. The fundamental solution of a linear partial differential equation with constant coefficients. *Dokl. Akad. Nauk SSSR (N.S.)* 119 (1958), 407-410. (Russian)

Let $L(\xi_1, \dots, \xi_m)$ be a homogeneous polynomial in ξ_1, \dots, ξ_m (with constant coefficients) such that all its first order partial derivatives vanish at no point other than $(0, \dots, 0)$. The fundamental solution K of

$$L(\partial/\partial x_1, \dots, \partial/\partial x_m)u = f(x_1, \dots, x_m)$$

(i.e., the solution of $LK = \delta(x_1, \dots, x_m)$) is analytic in m -dim. space except at points lying on the characteristic cone (consisting of such points (x_1, \dots, x_m) that $\sum x_i \xi_i = 0$ is tangent to $L(\xi_1, \dots, \xi_m) = 0$).

The author investigates the behavior of K in the neighborhood of C . Using the expression for K , the decomposition of δ as given by Gel'fand and Šapiro [Uspehi Mat. Nauk (N.S.) 10 (1955), no. 3(65), 3-70; MR 17, 371] and the Cauchy regularization (when L is non-elliptic), the author expands K in the power series of the distance s from the cone C , s being measured normally to C . The series are valid in the neighborhood of a point $p \in C$ at which K is regular and depends on m as well as on the local properties of C in the neighborhood of p .

C. Masaihs (Havre de Grace, Md.)

5966:

Mel'nik, D. P. Fundamental matrix of variational type systems for an unlimited space. *Dopovidi Akad. Nauk Ukrain. RSR* 1958, 602-605. (Ukrainian. Russian and English summaries)

This paper contains outlines of the proofs of a theorem on the existence of a fundamental matrix for an entire space and of a theorem of Liouville on an Euler system of partial differential equations.

H. P. Thielman (Ames, Iowa)

5967:

Fantappiè, Luigi. Costruzione generale delle soluzioni fondamentali delle equazioni a derivate parziali. Collect. Math. 9 (1957), 7-26.

L'A. applica la sua teoria degli operatori funzionali e dei funzionali analitici alla costruzione della soluzione fondamentale delle equazioni lineari a derivate parziali a coefficienti costanti. Egli considera l'equazione

$$(1) \quad \frac{\partial^m u}{\partial t^m} + \sum a_{r_0 r_1 \dots r_n} \frac{\partial^{r_0 + \dots + r_n} u}{\partial t^{r_0} \dots \partial x_n^{r_n}} = f(t, x_1, \dots, x_n)$$

$$(r_0 < m, r_0 + \dots + r_n \leq m)$$

e osserva, dapprima, che si può supporre $r_0 + \dots + r_n = m$, introducendo eventualmente la nuova variabile indipendente x_{n+1} e la nuova funzione incognita

$$\xi(t, x_1, \dots, x_n, x_{n+1}) = e^{x_{n+1} t} u(t, x_1, \dots, x_n).$$

Sia $t = \psi(x_1, \dots, x_n)$ la superficie (analitica) Γ portante i dati (analitici) di Cauchy, e sia S la famiglia delle funzioni $u_1(t, x_1, \dots, x_n)$ olomorfe in un intorno di Γ e nulle su Γ . L'A. introduce gli operatori I, B_r ponendo

$$If = \int_{\psi(x_1, \dots, x_n)} f(\tau, x_1, \dots, x_n) d\tau,$$

$$B_r f = \int_{\psi(x_1, \dots, x_n)} \frac{\partial f(\tau, x_1, \dots, x_n)}{\partial x_r} d\tau$$

e osserva che se $f \in S$ anche $B_r f \in S$; inoltre, in S , gli operatori B_r ($r=1, \dots, n$) sono permutabili. Si può perciò applicare la teoria degli operatori lineari dell'A. Applicando $m+1$ volte l'operatore I alla (1), si ottiene l'equazione

$$u_1 + \sum a_{r_0 r_1 \dots r_n} B_1^{r_1} B_2^{r_2} \dots B_n^{r_n} u_1 = If_2$$

con $u_1 \in S, If_2 \in S$. Posto $Q(B_1, \dots, B_n) = 1 + \sum a_{r_0 r_1 \dots r_n} B_1^{r_1} \dots B_n^{r_n}$, si ricava

$$(2) \quad u_1 = \frac{1}{Q(B_1, \dots, B_n)} If_2 = g(B_1, \dots, B_n) If_2$$

ove il valore $g(B_1, \dots, B_n) If_2$ è un funzionale G analitico e lineare della funzione $g(\lambda_1, \dots, \lambda_n)$:

$$g(B_1, \dots, B_n) If_2 = G(g(\lambda_1, \dots, \lambda_n)).$$

Tale valore si ottiene mediante un prodotto funzionale proiettivo:

$$(3) \quad G(g(\lambda_1, \dots, \lambda_n)) = \overset{\Delta \Delta}{\phi} g$$

ove ϕ è l'indicatrice proiettiva:

$$\phi(\alpha_1, \dots, \alpha_n) = G\left(\frac{1}{1 + \alpha_1 \lambda_1 + \dots + \alpha_n \lambda_n}\right)$$

$$= \frac{1}{1 + \alpha_1 B_1 + \dots + \alpha_n B_n} If_2.$$

Perciò ϕ è definita dall'equazione a derivate parziali del primo ordine a coefficienti costanti:

$$\frac{\partial \phi}{\partial t} + \alpha_1 \frac{\partial \phi}{\partial x_1} + \dots + \alpha_n \frac{\partial \phi}{\partial x_n} = f_2 \quad ((\phi)_\Gamma = 0).$$

Risulta allora

$$\phi(\alpha_1, \dots, \alpha_n, t, x_1, \dots, x_n) =$$

$$\int_{\Gamma} f_2(\tau, x_1 + \alpha_1(\tau - t), \dots, x_n + \alpha_n(\tau - t)) d\tau.$$

Una volta nota ϕ , il funzionale G si calcola con un numero finito di quadrature e di derivazioni mediante la (3): precisamente, posto

$$z = \frac{(n-1)!}{(1 + \alpha_1 B_1 + \dots + \alpha_n B_n)^n} If_2,$$

si ha

$$\overset{\Delta \Delta}{\phi} g = \frac{1}{(2\pi i)^n} \int_{C_1} d\alpha_1 \dots \int_{C_n} d\alpha_n \int_{\Omega} d\omega \frac{1}{\alpha_1 \dots \alpha_n}$$

$$\times g\left(-\frac{\tau_1}{\alpha_1}, \dots, \frac{\tau_{n-1}-1}{\alpha_n}\right) z(\alpha_1, \dots, \alpha_n),$$

ove le C_r sono curve chiuse fuori delle quali $g(-\tau_1/\alpha_1, \dots, (\tau_{n-1}-1)/\alpha_n)$ è analitica, e Ω è, nello spazio τ , la piramide che ha per vertice l'origine e i punti unitari sugli assi.

L. Amerio (Milan)

5968:

Vasilishin, S. A. The Cauchy problem in the class of operator-analytical functions. Dopovidi Akad. Nauk Ukrain. RSR 1958, 924-928. (Ukrainian. Russian and English summaries)

The following initial value problem for a function F of two variables u and x is considered: let M be a differential operator in u of order ν with continuous coefficients depending only on u , and L a differential operator in x of order n with the analogous properties. The problem is to find a solution F of the differential equation $MF = LF$ if F and its first $n-1$ derivatives with respect to x at $x=x_0$ are given functions of u which are M -analytic in the sense of M. K. Fage [Dokl. Akad. Nauk SSSR 112 (1957), 1008-1011; MR 20#1011a]. By the use of M -analytic functions this problem had been treated by Fage [Trudy Moscov. Mat. Obsč. 7 (1958), 227-268] in the "classical" case $n \geq \nu$.

The present paper treats the case $n < \nu$ with the main goal to generalize results known for (analytic) solutions of the equation $\partial^n z / \partial x^n = \partial^2 z / \partial u^2$. In this case the functions of u giving the initial conditions at $x=x_0$ must be entire functions of finite order ρ , and the discussion hinges on whether $\rho \leq \nu/4 - n$, and in case of the equality, on whether the given functions are of maximal, normal, or minimal type.

All this is carried over to the more general case at hand. In order to do so the author had first to define the notions of M -entire function and its order and type.

E. H. Rothe (Ann Arbor, Mich.)

5969:

Prokopenko, L. N. Cauchy problem for Sobolev's type of equation. Dokl. Akad. Nauk SSSR 122 (1958), 990-993. (Russian)

In a Hilbert Space H , let A and B_k be closed operators and consider the differential equation

$$\frac{d^m A U}{dt^m} + \sum B_k(t) \frac{d^k U}{dt^k} = f(t),$$

where k ranges from 0 to $m-1$. It is assumed that A and A^{-1} exist, have dense domains, but are not necessarily bounded. Moreover, $D(B_k) \supset D(A)$, and $B_k A^{-1}$ as well as $A^{-1} B_k$ are supposed bounded. After assuming appropriate weak continuity conditions, the author proves the existence of a unique solution $U(t)$ which is m -times weakly differentiable. The operator d/dt in the differential equation is understood to be the weak derivative, and $d^k U/dt^k = u_k$ holds for $t=0$ and $k=0, 1, \dots, m-1$, where $u_k \in D(A)$ are arbitrary initial values. In an interesting application the B_k are taken to be second order partial differential operators defined in E_n , and A is the closure of the Laplace operator acting at first on C_0^∞ . The hypotheses of the original problem are shown to be satisfied if the coefficients in B_k and their derivatives go to zero rapidly enough as $|x| \rightarrow \infty$.

A. N. Milgram (Minneapolis, Minn.)

POTENTIAL THEORY

See also 5961, 6125.

5970:

Berštein, I. Une caractérisation topologique de la pseudo-conjuguée d'une fonction pseudo-harmonique. *Z. Čist. Prikl. Mat.* 1 (1956), 51-54. (Russian)

A translation from the Romanian in Acad. R. P. Romîne. *Bul. Şti. Sect. Şti. Mat. Fiz.* 7 (1955), 75-78 [MR 17, 145]; a French translation was published in *Rev. Math. Pures Appl.* 1 (1956), no. 1, 45-48 [MR 19, 131].

5971:

Šamanskii, V. E. On harmonic functions in adjacent regions. *Ukrain. Mat. Ž.* 9 (1957), 329-343. (Russian. English summary)

"The basic result of this paper is the appraisal of the deviation of any harmonic function of two variables from two harmonic functions determined in adjacent regions, the sum of which is the initial region. The values of the harmonic functions coincide on the border of the initial region. Analogous appraisals are presented in the paper for conjugate functions and for solutions of a two-dimensional Poisson's equation." *Author's summary*

5972:

Kondurar', V. T. The expansion of the potential function of the mutual attraction of two ellipsoids (homogeneous and non-homogeneous). *Astr. Ž.* 35 (1958), 763-771. (Russian)

A multiple series is found for the mutual potential of two spheroids, one of which is homogeneous, arbitrarily situated relative to one another. An earlier work of the author was restricted to a special orientation of the bodies [Ivanov. *Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauk.* 5 (1954), 103-115; MR 17, 1074].

F. V. Atkinson (Canberra City)

5973:

Tovmasyan, N. E. Existence of a solution of the Dirichlet problem for the Laplace equation in the case of non-symmetric boundary values. *Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk* 11 (1958), no. 3, 23-40. (Russian. Armenian summary)

Let D be a domain in three-dimensional space for which the Dirichlet problem with continuous boundary data is soluble. Let P_0 be a point on the boundary σ of D and such that σ has a continuously varying normal near P_0 and can be represented by an equation $z=F(x, y)$ near P_0 (z -axis in the direction of the normal at P_0). The author proves that for every function $F(P)$ defined and continuous on $\sigma-P_0$ there is a function $U(P)$ harmonic in D , continuous in $D+\sigma-P_0$ such that, on $\sigma-P_0$, $U(P)=F(P)$. The proof is based on the construction of a function $V(P)$ harmonic in D , continuous on $D+\sigma-P_0$ such that $V(P)-F(P)$ is bounded in $D+\sigma-P_0$. As an application the author finds conditions for the solubility of the two-dimensional Dirichlet problem with boundary data which are continuous except at one point.

W. H. J. Fuchs (Ithaca, N.Y.)

5974:

Patraulea, N. N. Sur la solution du problème de Prandtl généralisé. I. *Acad. R. P. Romîne Stud. Cerc. Mec. Apl.* 9 (1958), 525-536. (Romanian. Russian and French summaries)

"On entend par problème de Prandtl généralisé la dé-

termination d'une fonction harmonique ϕ , régulière à l'extérieur des frontières $D+D_1$ et à l'infini et satisfaisant sur D (la frontière D étant une portion d'une surface ouverte, dans le cas de la fonction ϕ de trois variables, et un arc de courbe, dans le cas de la fonction ϕ de deux variables) aux conditions $d\phi^+/dn=d\phi^-/dn$ et $\phi(P^+)-\phi(P^-)=F(P)+G(P)d\phi/dn$. Sur les frontières D , la dérivée normale doit prendre des valeurs données.

On cite des applications particulières du problème de Prandtl et l'on montre une analogie entre le problème du profil à jet de bord de fuite et le problème de l'aile d'envergure semi-infinie.

Le problème général est réduit à la solution d'une équation intégrale de Fredholm, sur le domaine D , seulement. On étudie les propriétés du noyau et on indique les méthodes de solution, numériques ou basées sur l'analogie électrique. Des cas concrets seront étudiés en détail dans le prochain numéro de cette publication."

Résumé de l'auteur

5975:

Yamasuge, Hiroshi. Harmonic functions with two singular points. II. *J. Inst. Polytech. Osaka City Univ. Ser. A.* 8 (1957), 181-183.

[For part I, see same *J. Ser. A* 8 (1951), 39-42; MR 19, 405.] The present part proves three theorems about the harmonic function $\phi(x)$ satisfying the equation discussed in part I. These three theorems are essentially summed up in the second of them, namely: if ψ is an arbitrary harmonic function having two singular points ξ_1 and ξ_2 and satisfying the same equation as ϕ , then ψ can be written as $a\phi+b$, where a and b are constants.

5976:

Bergman, Stefan. Applications of function theoretical methods in the study of harmonic functions and vectors of three variables. *Ann. Acad. Sci. Fenn. Ser. A. I.* no. 250/2 (1958), 10 pp.

The author sketches briefly a method for expressing a harmonic function of three (real) variables as an integral transform of an analytic function of two complex variables; then indicates how certain properties of the latter function are reflected in the former. Similarly, he studies the representation of harmonic (=solenoidal and irrotational) vector fields as integral transforms of analytic functions. Finally, it is indicated that similar methods may be used to generate solutions of certain types of linear elliptic equations.

Bernard Epstein (Philadelphia, Pa.)

5977:

di Palo, Raffaele. Sul problema al contorno per le funzioni biarmoniche. *Giorn. Mat. Battaglini* (5) 5(85) (1957), 296-304.

The author derives the Hadamard variational formula for the first variation of the Green's function associated with the iterated Laplace operator Δ^2 for compact domains in space, bounded by a surface of class C^2 .

E. Calabi (Princeton, N.J.)

5978:

Zolin, A. F. An approximate solution of the polyharmonic problem. *Dokl. Akad. Nauk SSSR* 122 (1958), 971-973. (Russian)

Considers method of obtaining least-squares approximations to boundary conditions by a linear combination of polyharmonic functions, proving convergence of the method.

A. S. Householder (Oak Ridge, Tenn.)

5979:

*Spencer, D. C. A spectral resolution of complex structure. Symposium internacional de topologia algebraica [International symposium on algebraic topology], pp. 68-76. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

The notion of an \mathcal{M} -structure on X , where X is a topological space and \mathcal{M} is a local category [Cartan and Eilenberg, same Symposium, pp. 16-23; MR 20 #4837] is introduced. The definitions of almost-complex and complex structure on a manifold are given in these terms.

The purpose of the paper is to introduce a bigradation of harmonic functions on a hermitian manifold with boundary which generalizes the bigradation of harmonic polynomials in complex-euclidian space. Let V be an open submanifold of a hermitian almost-complex manifold X and assume that bV , the boundary of V , is a compact regularly imbedded submanifold of X . Also assume that the almost-complex structure is integrable in a neighborhood of V . Let H be the Hilbert space of norm-finite functions on bV and let $G: H \rightarrow H$ be the operator defined by

$$(G\varphi)(y) = \int_{bV} \overline{g(x, y)} \varphi(x) dS_x,$$

where dS_x is the volume element on bV and $\overline{g(x, y)}$ is the kernel of the Green's operator for some finite manifold W such that $X \supset W \supset V$. G is a symmetric completely continuous injective transformation whose range consists of boundary values of harmonic functions on V . Let S be the space of harmonic functions on V whose boundary values are in $G(H)$; we can interpret G also as a transformation of H onto S . Let T be the transformation defined by $T = G \circ \partial \bar{\partial} G$, where $\bar{\partial}$ is the anti-holomorphic component of the exterior derivative $\bar{\partial}$ and where $\bar{\partial}$ maps forms defined on V into their normal components on bV . Now for each positive number τ the author defines the operator $d_\tau = \tau \bar{\partial} + \bar{\partial}$. To d_τ we can associate an operator $T_\tau: H \rightarrow S$ analogous to the operator T above. We can interpret the transformation T as a map of H into H by restricting the functions in S to H . Let λ_τ be the eigenvalues of T_τ ; then we have the direct sums $S_\tau = \sum S_{\lambda_\tau}$ and also $S = \sum S_{\lambda_\tau}$ where the λ are eigenvalues of T . We say that f and g in S are equivalent if for each sufficiently small τ there exist functions f_τ, g_τ in S_{λ_τ} such that f_τ, g_τ converge to f, g respectively. In this way to each equivalence class in S_λ (and small enough τ) is associated a unique λ_τ and $\lim \lambda_\tau = \lambda$. Now to each equivalence class we associate (r, s) as follows: $s = \lambda$; r is defined as $c_t - \lambda$ if $\lambda_\tau - \lambda = c_t \tau^t + O(\tau^{t+1})$ with $c_t \neq 0$ and, if no such c_t exists, as $-\lambda$; thus the author obtains the desired bigradation of S , namely $S = \sum S_{r,s}$, where $S_{r,s}$ is the subspace of functions whose equivalence classes are associated with (r, s) .

The author concludes with some remarks about the behaviour of these spaces under a deformation of the almost-complex structure on V .

J. J. Kohn (Waltham, Mass.)

5980:

Gaffney, Matthew P. Asymptotic distributions associated with the Laplacian for forms. Comm. Pure Appl. Math. 11 (1958), 535-545.

Let M be a compact, n -dimensional, Riemannian manifold of class C^∞ . The author investigates the asymptotic distribution of the characteristic forms and values of the Laplace-Betrami operator $\Delta = d\bar{\partial} + \bar{\partial}d$ which acts on p -forms on M . The method used is to study the heat operator $\Delta + \partial/\partial t$. The fundamental solution of the heat equation is given in terms of the characteristic p -forms ω_i of Δ by

$$\theta(P, Q; t) = \sum_{i=1}^{\infty} \omega_i(P) \omega_i(Q) e^{-\lambda_i t},$$

where the λ_i are characteristic values of Δ . The author finds the asymptotic behavior of θ and, by applying Karamata's Tauberian theorem, he obtains the following asymptotic distributions:

$$\lambda_k \sim 4\pi \left[\frac{\Gamma(\frac{1}{2}n+1)k}{\binom{n}{p} V} \right]^{2/n}, \quad \sum_{k=1}^k |\omega_k(P)|^2 \sim K/V.$$

J. J. Kohn (Waltham, Mass.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

5981:

Jury, Eliahu I.; and Mullin, Francis J. A note on the operational solution of linear difference equations. J. Franklin Inst. 266 (1958), 189-205.

Elementary questions in the theory of linear difference equations with constant coefficients (and other equations closely related to these) are treated by a method which, apart from trivial notational changes, is the classic method of generating functions.

W. Strod (New York, N.Y.)

5982:

Hahn, Wolfgang. Über die Anwendung der Methode von Ljapunov auf Differenzengleichungen. Math. Ann. 136 (1958), 430-441.

The author investigates the stability of solutions of difference equations of the form 1) $x(t+1) = f(x, t)$, by means of a modification of Ljapunov's second method for differential equations. The real independent variable t is restricted to the sequence of values t_0, t_0+1, \dots , for which the initial point t_0 itself is chosen from a specified sequence t^*, t^*+1, \dots . The dependent variable x is an n -row column vector with components x_1, \dots, x_n in a region $|x| \leq h$, and $f(x, t)$ is a column vector with components $f_i(x, t) = f_i(x_1, \dots, x_n, t)$ bounded in a suitable region, continuous with respect to the x_i 's and such that $f_i(0, \dots, 0, t) = 0$. Equation 1) thus possesses the trivial solution $x=0$. The author defines "stability", "asymptotic stability", and "instability" for the trivial solution of 1). In a number of theorems he relates these types of behavior of the trivial solution of 1) to the defined properties "positive definite" and "positive definite decreasing" of an associated "Ljapunov function" $v(x, t)$ and the function $w(x, t) = v(x(t+1), t) - v(x, t)$.

If eq. 1) has the particular form 2) $x(t+1) = Ax(t)$, where A is a matrix with real, constant elements, the author shows that the trivial solution of 2) is either asymptotically stable or unstable according as the characteristic values of the matrix A are, respectively, all less than unity in absolute value or not. The concept of asymptotic stability is further refined by definitions of "exponential stability" and "total exponential instability". For eq. 1) in the form 3) $x(t+1) = A(t)x(t)$, where $A(t)$ is a bounded, non-singular matrix, the exponential stability or total exponential instability of the trivial solution of 3) is related to the existence and the determination of the respective properties of the associated Ljapunov function $v(x, t)$ and the function $w(x, t)$.

If eq. 1) has the form 4) $x(t+1) = A(t)x(t) + g(x(t), t)$, where $g(0, t) = 0$ and $|g(x(t), t)| \leq a|x(t)|$ for suitable a independent of t , it is shown that for suitably small a , the trivial solutions of 3) and 4) are either both exponentially stable or both totally exponentially unstable. The paper concludes with brief references to related problems concerning differential equations.

P. E. Guenther (Cleveland, Ohio)

5983:

Forder, H. G. Duplication formulae. Math. Gaz. 41 (1957), 215-217.

The author observes that the cosine must have zeros because $C(x) = 1$ is the only strictly positive solution for (*) $C(2x) = 2C^2(x) - 1$ satisfying $C(x) \leq 1$. He shows once again that $\cos kx$ and $\cosh kx$ are the only real-valued, non-constant, continuous, even solutions for (*) that are twice differentiable at $x=0$. [For other settings and earlier work see Fenyő, Acta Math. Acad. Sci. Hungar. 7 (1956), 383-396; MR 19, 152; Vaughn, Amer. Math. Monthly 62 (1955), 707-713; MR 17, 631; and Vietoris, J. Reine Angew. Math. 186 (1944), 1-15; MR 6, 271.]

G. Crane (Pittsburgh, Pa.)

5984:

Cooper, R. On a duplication formula. Math. Gaz. 41 (1957), 217-218.

The note describes real, even, continuous solutions for $\phi(2x) = 2\phi^2(x) - 1$ that are differentiable once but not necessarily twice at $x=0$ [see preceding review]. An explicit example shows that additionally requiring $\phi(x)$ to be convex does not lead to the solution $\cosh kx$.

G. Crane (Pittsburgh, Pa.)

SEQUENCES, SERIES, SUMMABILITY

5985:

Szűsz, Péter. A problem in the theory of series. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 6 (1956), 461-465. (Hungarian)

5986:

Rényi, Kató. On an infinite system of linear equations. Mat. Lapok 8 (1957), 61-67. (Hungarian. English and Russian summaries)

If

$$\sum_{n=1}^{\infty} n^{2k-1} x_n = b_k \quad (k=1, 2, 3, \dots)$$

holds with all $b_k = 0$ and not all $x_n = 0$, then $\limsup |x_n|^{1/n} \geq 1$. If $b_k = 0$, where $k_s < cs$, $c > 1$, and not all $x_n = 0$, then $\limsup |x_n|^{1/n} > 0$.

G. Szegő (Stanford, Calif.)

5987:

Benneton, Gaston. Sur les produits infinis complexes semi-convergenes. C. R. Acad. Sci. Paris 247 (1958), 1284.

If for some integer $k \geq 2$ the $k-1$ series $\sum_{n=1}^{\infty} u_n^p$ ($p=1, 2, \dots, k-1$) are convergent and the series $\sum_{n=1}^{\infty} u_n^k$ is absolutely convergent, then the infinite product $\prod_{n=1}^{\infty} (1+u_n)$ is convergent.

A. G. Aspinia (Amherst, Mass.)

5988:

Negoescu, N. Valeurs moyennes symétriques des fractions continues de la forme

$$\theta_i = [a_0, a_1, a_2, \dots, a_r, a_{r+1}, i] \quad (i=1, 2, 3, \dots, m).$$

Gaz. Mat. Fiz. Ser. A (N.S.) 10(63) (1958), 482-491. (Romanian. French and Russian summaries)

A proof is given of the theorem: Let

$$\theta_i = [a_0, a_1, a_2, \dots, a_r, a_{r+1}, i] \quad (i=1, 2, 3, \dots, m)$$

be m continued fractions; then

$$\frac{\sum \theta_i}{m} = [a_0, a_1, a_2, \dots, a_r, b_{r+1}],$$

where b_{r+1} satisfies the inequalities

$$\frac{m \cdot a_{r+1,1} \cdots a_{r+1,m}}{\sum a_{r+1,1} \cdots a_{r+1,m-1}} \leq b_{r+1} \leq \frac{\sum a_{r+1,i}}{m}.$$

This is a generalization of a theorem of R. Robinson [Bull. Amer. Math. Soc. 54 (1948), 693-705; MR 10, 235].

E. Frank (Chicago, Ill.)

5989:

Erwe, Friedhelm. Axiomatische Fragen der Limitierungstheorie. Bonn. Math. Schr. no. 4 (1957) 70 pp.

5990:

Mayer, Daniel; und Nečas, Jindřich. Das Addieren unendlicher Reihen unter Benützung von Integraltransformationen. Apl. Mat. 1 (1956), 165-185. (Czech. Russian and German summaries)

Im Artikel ist die Operatorenmethode der Addition von unendlichen konvergenten Reihen beschrieben. Einfachheit und ein breites Applikationsgebiet sind wesentliche Vorteile dieser Methode.

Die Hauptidee, auf welcher die beschriebene Methode beruht, ist aus dem folgenden Beispiel ersichtlich, in dem wir die relativ konvergente Reihe

$$(1) \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

addieren. Im Sinne der Laplace'schen Transformation suchen wir ein solches Bild $F(p)$, damit $F(n) = 1/n$. Nachdem uns bekannt ist, dass $\int_0^{\infty} 1 \cdot e^{-pt} dt = 1/p$, ist dies nicht schwer. Die Reihe kann als Summe von Werten der Funktion $F(p)$ für $p=1, 2, \dots$, mit zugehörigen Abzeichen angesehen werden. Drücken wir die Reihe (1) als Summe der Bilder in einem Punkte $p=1$ aus:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} F(1+(n-1)).$$

Das zugehörige Bild ist also

$$\sum_{n=1}^{\infty} (-1)^{n-1} F(p+(n-1)) = G(p),$$

wo $G(p)$ das Bild der Funktion $g(t)$ ist. Dann

$$(2) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \int_0^{\infty} g(t) e^{-t} dt,$$

$$L\{(-1)^{n-1} e^{-(n-1)t}\} = (-1)^{n-1} F(p+(n-1)),$$

so dass der Summe der Bilder die Summe der Originale

$$\sum_{n=1}^{\infty} (-1)^{n-1} e^{-(n-1)t}$$

entspricht. Im gegebenen Falle wird also die Addition

der Reihe in die Berechnung des Integrals (2) überführt:

$$\sum_{n=1}^{\infty} (-1)^{n-1} e^{-(n-1)t} = \frac{1}{1+e^{-t}},$$

und also

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \int_0^{\infty} \frac{e^{-t}}{1+e^{-t}} dt = \int_0^{\infty} \frac{1}{1+e^t} dt = \lg 2.$$

Ausser der Laplace'schen Transformation kann analogisch die Fourier'sche Transformation benutzt werden. Im übrigen Teil des Artikels wird die Frage der Verwendung der Mellin'schen inversen Transformation behandelt. Diese Methode wird durch eine Reihe von Beispielen illustriert.

Aus der Zusammenfassung der Autoren

5991:

Copping, J. Inclusion theorems for conservative summation methods. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 485-499.

The author considers conservative matrices A (which transform every convergent sequence into a convergent sequence) and their inverses. A conservative A is called coregular if

$$\chi(A) = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk} \neq 0;$$

otherwise conull. As usual, $\|A\| = \sup_n \sum_k |a_{nk}|$. He obtains many results, some of them with quite delicate proofs. Theorem 1: If C is coregular, if $\|B\| < \infty$, and if $A=BC$ is conservative, then there exists a conservative E such that $A=EC$. (This answers a question of Wilansky and Zeller [J. London Math. Soc. 32 (1957), 397-408; MR 19, 646].) The proof depends upon the fact that if $\|B\| < \infty$ one can select a sub-matrix with existing column-limits. By subtracting such limit-rows from B the author constructs the E in question. (From the general point of view the problem is as follows: B maps a closed subspace of (c) , the set of all convergent sequences, into (c) ; can this mapping be extended to the whole space (c) ?) Theorem 3: If C is conull, and if A is coregular, then C sums a bounded sequence which is not summed by A .

The following theorems deal with reversible matrices C (those which have inverse mappings of the form $x_n = \sum_k \lim y_k + \sum_k g_{nk} y_k$). Using this representation the author proves inclusion and consistency theorems for reversible matrices, e.g., theorem 9: Suppose that C is reversible and coregular, and that A is not weaker than C . Then A is consistent with C if and only if AG is regular. — Hence a C of type M is consistent with every stronger A which has the same column- and row sum-limits. — Counterexamples show that theorem 1 is not valid if we allow C to be conull, and that the equation $A=(AG)C$ (where C is reversible) is not always true. *K. Zeller* (Tübingen)

5992:

Meder, J. Application of Mazur's theorem on convergence multipliers to sequences limitable by the Euler-Knopp method. Prace Mat. 2 (1958), 329-336. (Polish. Russian and English summaries)

The author considers the Euler-Knopp method (E, q) (where $q > 0$) which is defined by the sequence-to-sequence transform

$$y_n = (q+1)^{-n} \sum_{k=0}^n \binom{n}{k} q^{n-k} x_k.$$

Numbers c_n are called convergence factors for (E, q) if the series $\sum c_n u_n$ converges for every (E, q) -limitable sequence $\{u_n\}$. Using a general convergence factor theorem of

Mazur [Math. Z. 28 (1928), 599-611; p. 602] he proves that

$$\limsup |c_n|^{1/n} < 1/(2q+1)$$

is a sufficient condition for convergence factors; while

$$\limsup |c_n|^{1/n} \leq 1/(2q+1)$$

is necessary, as shown by a power series example. Both results follow more easily from the fact that $u_n = o((2q+1)^n)$ is the exact order condition for (E, q) , or from function theoretic considerations. [For more literature see the reviewer's book "Theorie der Limitierungsverfahren", Springer, Berlin-Göttingen-Heidelberg, 1958; MR 20 #4119; p. 130, 133 and 156.] *K. Zeller* (Tübingen)

5993:

Topuriya, S. B. On some theorems of Tauberian type for double series. Soobsč. Akad. Nauk Gruzin. SSR 20 (1958), 129-136. (Russian)

5994:

Shanks, Daniel. Two theorems of Gauss. Pacific J. Math. 8 (1958), 609-612.

The author obtains the identity

$$\prod_{i=1}^{\infty} \frac{1-x^{2^i}}{1-x^{2^{i-1}}} = \sum_{i=1}^{\infty} x^{i(i-1)},$$

as well as the evaluation of Gauss' sum from the identities

$$\sum_{s=0}^{n-1} \frac{P_n}{P_s} x^{s(2n+1)} = \sum_{s=1}^{2n} x^{i(i-1)},$$

$$\sum_{s=1}^n \frac{P_n}{P_s} x^{s(2n+1)} = \sum_{s=1}^{2n+1} x^{i(i-1)},$$

where $P_n = \prod_{s=1}^{n-1} (1-x^{2^s})/(1-x^{2^{s-1}})$, stated in an earlier paper [Proc. Amer. Math. Soc. 2 (1951), 747-749; MR 13, 321; see p. 749]. The latter identities are proved in the present paper. *L. Carlitz* (Durham, N.C.)

5995:

Hlawka, Edmund. Folgen auf kompakten Räumen. II. Math. Nachr. 18 (1958), 188-202.

For part I see Abh. Math. Sem. Univ. Hamburg 20 (1956), 223-241 [MR 18, 390]. The present part II continues with 17 theorems of the sort described in the review of part I.

APPROXIMATIONS AND EXPANSIONS

See also 5877, 5892, 6187.

5996:

von Targonski, Georg. Interpolation durch Reihen iterierter Funktionen. Soc. Sci. Fenn. Comment. Phys.-Math. 20 (1957), no. 9, 12 pp.

Pour une fonction $\varphi(n)$ de la variable réelle x , écrite sous la forme $\varphi(x) = h c h^{-1}(x)$, où c désigne une constante, h^{-1} la fonction inverse de h , on a: $\varphi^n = h c^n h^{-1}$. Prenant $h = \sin x$, $c = 4$, on a: $\varphi^n(x) = \sin(4^n \arcsin x)$, puis posant: $\alpha_n = \pi/2 \cdot 4^n$ ($n=0, 1, \dots$), l'auteur montre qu'une fonction $F(x)$, prenant aux points α_n des valeurs $\xi_n = O(1/R^n)$, avec $R > 1$, continue dans le segment $[-1, +1]$, est donnée par: $F(x) = \sum_{n=0}^{\infty} a_n \varphi^n(x)$, où les a_n sont les coefficients du développement de Taylor de la fonction: $\sum_{n=0}^{\infty} \xi_n x^n = \sum_{n=0}^{\infty} a_n x^n$. Généralisation au cas où $c=6, 8, \dots$

J. Favard (Paris)

5997:

Gould, H. W. A theorem concerning the Bernstein polynomials. *Math. Mag.* 31 (1957/58), 259-264.

If $f(x)$ is a polynomial of degree r and $B_n(x)$ is the Bernstein polynomial of f of order n , then it is easy to see that

$$B_n(x) - f(x) = \sum_{s=1}^r n^{-s} Q_{sr}(x),$$

where Q_{sr} are polynomials not depending on n . [Compare G. Lorentz, Bernstein polynomials, Univ. of Toronto Press, 1953; MR 15, 217; p. 22.] The author finds the coefficients of the Q_{sr} . G. G. Lorentz (Syracuse, N.Y.)

5998:

Talalyan, A. A. On convergence in measure of series in bases of the space L_p . *Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk* 10 (1957), no. 1, 31-68. (Russian. Armenian summary)

En généralisant un théorème de Men'sov [Trudy Mat. Inst. Steklov 32 (1950); Amer. Math. Soc. Transl. no. 105 (1954); MR 12, 254; 15, 866] concernant les séries trigonométriques, l'auteur démontre le théorème suivant: soit $\{\varphi_n(x)\}$ un système de fonctions définies sur un ensemble mesurable $GC[0, 1]$, mes $G > 0$, et formant une base normée sur $L_p(G)$, $p > 1$. A toute fonction mesurable f définie sur G correspond une série $\sum a_n \varphi_n(x)$ qui converge en mesure sur G vers $f(x)$, et $\lim a_n = 0$. Il existe aussi une suite $\{b_n\}$, avec les b_n non tous nuls, tels que $\sum b_n \varphi_n(x)$ converge en mesure sur G vers zéro.

S. Mandelbrojt (Paris)

5999:

Freud, G. Eine Bemerkung zur asymptotischen Darstellung von Orthogonalpolynomen. *Math. Scand.* 5 (1957), 285-290.

Soit $f(\theta) \in L$ une fonction non négative de période 2π , telle que: $\log f(\theta) \in L$. Appelant $k(f, z)$ la fonction holomorphe dans le cercle unité, telle que: $k(0) > 0$, et que $\Re[k(re^{i\theta})] \rightarrow f(\theta)$ presque partout lorsque r tend vers 1 (\Re désigne la partie réelle); posant: $D(f, z) = \exp(\frac{1}{2}k)$, puis définissant la suite de polynômes orthogonaux: $\phi_n(f, z) = K_n z^n + \dots$, par les conditions:

$$K_n > 0, \int_{-\pi}^{+\pi} |\phi_n|^2 / (2) d\theta, \int_{-\pi}^{+\pi} \phi_n(f, e^{i\theta}) \Pi(e^{i\theta}) / (f(\theta)) d\theta = 0,$$

où $\Pi(z)$ désigne un polynôme quelconque de degré plus petit que n , et où la barre indique le passage au nombre complexe conjugué, on sait que, sous certaines conditions exposées dans Szegő [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. vol. 23, New York, 1939; MR 1, 14], on a:

$$(1) \quad \phi_n[f, e^{i\theta}] = e^{in\theta} \{D(f, e^{i\theta})\}^{-1} + o(1).$$

L'auteur démontre que si f est à variation bornée et s'il existe deux constantes m et M telles que: $0 < m \leq f \leq M$, alors $\phi_n = O(1)$ uniformément en θ et n ; si, de plus:

$$(2\pi)^{-1} \int_{-\pi}^{+\pi} \log f(\theta) \cotg \frac{1}{2}(\gamma - \theta) d\theta$$

existe au sens de la valeur principale de Cauchy, alors l'égalité (1) vaut pour $\theta = \gamma$. J. Favard (Paris)

6000:

Gapoškin, V. F. A generalization of the theorem of M. Riesz on conjugate functions. *Mat. Sb. N. S.* 46(88) (1958), 359-372. (Russian)

The author seeks conditions on ϕ which allow an inequality

$$\int_{-\pi}^{\pi} |\tilde{f}(x)|^p \phi(x) dx \leq A \int_{-\pi}^{\pi} |f(x)|^p (\phi(x)) dx \quad (p > 1),$$

where \tilde{f} is the function conjugate to f in the sense used in Fourier series. The case $\phi(x) = x^\alpha$ was treated by Babenko [Dokl. Akad. Nauk SSSR 62 (1948), 157-160; see MR 10, 249]. The author shows that this is the case if $\phi \in L$, $\phi(x) > 0$ almost everywhere, and

$$(*) \quad \phi(r, x) \geq c |\psi(r, x)|$$

(where $\phi(r, x)$ is the Poisson integral of ϕ and ψ is its harmonic conjugate), with $c > 0$ when $p \geq 2$, $c > |\tan \frac{1}{2} p \pi|$ when $1 < p \leq 2$. He uses this result to discuss a problem suggested by Bary. Consider a normalized basis in Hilbert space H and its conjugate basis $\{y_k\}$. The basis has the Bessel property if $\sum |(x, y_k)|^2$ converges for all x ; it has the Hilbert property if whenever $\sum c_k^2 < \infty$ there is an x in H such that $(x, y_k) = c_k$; it has the Riesz property if it has both the Bessel and Hilbert properties. Babenko showed by using his results cited above that a basis does not have to have either the Bessel or the Hilbert property. The author considers the system $\{(2\pi)^{-1} t(x) e^{i n x}\}$. If $\phi(x) = t^2(x)$ satisfies (*) with $c > 0$, this system is a basis and remains so if t is replaced by $1/t$. If $0 < m \leq M < \infty$, then (inequalities supposed to hold almost everywhere) (1) if $m \leq |t(x)| \leq M$, the system has the Riesz property; (2) if $m \leq |t(x)|$ but $|t|$ is unbounded above, the system has the Bessel property but not the Hilbert property; (3) if $|t(x)| \leq M$ but $1/|t|$ is unbounded above then the system has the Hilbert property but not the Bessel property; (4) if neither $|t|$ nor $1/|t|$ is bounded then the basis has neither the Hilbert nor the Bessel property.

R. P. Boas, Jr. (Evanston, Ill.)

6001:

Zuhovickii, S. I.; and Ėskin, G. I. Čebyšev approximation in a Hilbert ring. *Dokl. Akad. Nauk SSSR (N.S.)* 118 (1958), 870-872. (Russian)

Let H be a Hilbert ring (H^* -algebra in W. Ambrose's terminology [Trans. Amer. Math. Soc. 57 (1945), 364-386; MR 7, 126]), Q a compact metric space, and φ a norm-continuous mapping of Q into H . The problem of Čebyšev approximation of a continuous H -valued function f on Q by function $a\varphi$ ($a \in H$) is to find an element $a^{(0)} \in H$ such that

$$\max_{q \in Q} \|a^{(0)} \varphi(q) - f(q)\| = \inf_{a \in H} \max_{q \in Q} \|a \varphi(q) - f(q)\|.$$

This problem is a special case of one treated earlier by the first-named author [Mat. Sb. N.S. 37(79) (1955), 3-20; MR 17, 388]. Theorem 1: An element $a^{(0)}$ as above exists for every continuous f on Q if and only if the smallest right ideal in H containing $\varphi(Q)$ is a direct sum $p_1 H \oplus \dots \oplus p_k H$ ($p_1, \dots, p_k \in H$). Theorem 2: Let φ satisfy the condition of Theorem 1. Then the element $a^{(0)}$ is unique if and only if $a\varphi(q) = \theta$ implies $a = \theta$ for all $q \in Q$. No proofs are given. See also a previous note by the authors [Doklady Akad. Nauk SSSR 116 (1957), 731-734; MR 20 #2610]. E. Hewitt (Seattle, Wash.)

6002:

Stečkin, S. B. Approximation of abstract functions. *Rev. Math. Pures Appl.* 1 (1956), no. 3, 79-83. (Russian)

This note discusses the problem of best uniform approximation of various Banach space-valued functions defined on a compact metric space Q . It is closely connected with two papers by S. I. Zuhovickii and the author [Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 385-388, 773-776; MR 18, 222; 19, 30]. The new results announced generalize those announced loc. cit. and are related to those announced in the paper reviewed above. Let X and Y be Banach spaces, and let $F(t)$ ($t \in Q$) be a family of

bounded linear operators on X into Y such that for every $x \in X$, $F(t)x$ is a norm-continuous map of Q into Y . For a given norm-continuous map f of Q into Y , one may look for an element $x_f \in X$ such that

$$\max_{t \in Q} \|f(t) - F(t)x_f\| = \inf_{x \in X} \max_{t \in Q} \|f(t) - F(t)x\|.$$

Existence and uniqueness of x_f are discussed under a number of restrictions on X , Y , and $F(t)$. No proofs are given.
E. Hewitt (Seattle, Wash.)

6003:

*de Bruijn, N. G. *Asymptotic methods in analysis*. Bibliotheca Mathematica. Vol. 4. North-Holland Publishing Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publishers Inc., New York; 1958. xii+200 pp. \$5.75.

"This book arose from a course of lectures ... Its purpose is to teach asymptotic methods by explaining a number of examples in every detail, so as to suit beginners who seriously want to acquire some technique in attacking asymptotic problems ... This book has not been written exclusively for mathematicians, but also for those physicists and engineers who have a certain maturity ... This is not an encyclopaedia on asymptotic results. Not even the asymptotic behaviour of Bessel functions can be found in this book. Attention is focussed mainly on methods."

These few excerpts from the author's preface indicate the character of de Bruijn's book. A prospective reader looking for theorems of utmost generality or for concise statements of wide application will be as much disappointed as one looking for a survey of the whole field; but a reader looking for interesting problems tackled often by highly original methods, for precise results fully proved, and for procedures fully motivated will be delighted — although, in the reviewer's opinion, even such a reader may find the book almost too conversational in places.

Contents. Chapter 1. Introduction. The definition and elementary properties of asymptotic expansions. Chapter 2. Implicit functions. Both analytic and asymptotic inversion are discussed here, as is asymptotic iteration. Chapter 3. Summation. Various examples are discussed under several headings according as the main contribution to the sum comes from a single term or a small group of terms, and according as these occur at the beginning or the end, or else somewhere in the middle, of summation. Applications of the Euler-Maclaurin and Poisson sum formulas are given here. Chapter 4. The Laplace method for integrals. Includes a discussion of multiple integrals. Chapter 5. The saddle point method. General description and some simpler examples. Chapter 6. Applications of the saddle point method. This chapter contains three difficult examples: (i) the number of partitions of a finite set into disjoint non-empty subsets; (ii)

$$S(s, n) = \sum_{k=0}^{2n} (-1)^{k+n} \binom{2n}{k}^s,$$

where s is a fixed positive integer and $n \rightarrow \infty$; (iii)

$$\int_0^\infty e^{-P(u)} u^{s-1} du,$$

where $P(u)$ is a polynomial and s a large complex variable. Chapter 7. Indirect asymptotics. This chapter contains some Tauberian theorems. Chapter 8. Iterated functions. Here, f is a given function, $x_n = f(x_{n-1})$, $n=1, 2, \dots$, and the behaviour of x_n as $n \rightarrow \infty$ is studied in several interesting examples. This is one of the most ingenious chapters of this ingenious book. Chapter 9. Differential

equations. The Riccati equation

$$t^{-k} p'(t) = \alpha(t) + \beta(t)p(t) + \gamma(t)p^2(t)$$

is studied as $t \rightarrow \infty$, and the results are applied to second order linear differential equations.

[In a book of this character the choice of the material is a function of the author's taste, and a function, at that, which (in de Bruijn's terminology) behaves rather violently. The reviewer is expressing his personal taste rather than an objective appraisal of the author's choice in saying that he finds 23 pages on differential equations, versus 74 pp. on Laplace's method and the saddle point method, somewhat inadequate; that he regrets the complete absence of the method of stationary phase; and that he looked in vain for an adequate reference to van der Corput's important work in this field.]

This is a highly personal, ingenious, stimulating, and in parts even exhilarating book. It has enough original material to appeal to the expert. It would be an excellent book to read in conjunction with a textbook or monograph on asymptotic expansions — if such a one existed: whether it can replace a textbook is a question which every reader will have to answer for himself.]

A. Erdélyi (Pasadena, Calif.)

6004:

Arnol'd, G. A. *On an asymptotic formula obtained in approximating continuous functions by singular integrals of a special form*. Ukrain. Mat. Ž. 10 (1958), 328-333. (Russian)

The author obtains the principal term, the first order term, and the order of magnitude of the next term, in the asymptotic expansion of the integral

$$\int_a^b k(x, t) f(t) dt$$

as $n \rightarrow \infty$, under suitable assumptions of $k(x, t)$ and $f(t)$.

R. Bellman (Santa Monica, Calif.)

FOURIER ANALYSIS

See also 5953.

6005:

Devinatz, A.; and Hirschman, I. I., Jr. *The spectra of multiplier transforms* l_p . Amer. J. Math. 80 (1958), 829-842.

Let $T = (t_{j-k})$, where $j, k=0, \pm 1, \dots$, be a Toeplitz matrix. Let $I(T)$ be the set of p -values, $1 \leq p \leq \infty$, for which T is a bounded operator from l_p to l_p and let $S_p(T)$ be the spectrum of this operator. Then $I(T)$ is empty or is an interval (open or closed) with endpoints μ, ν , where $1/\mu + 1/\nu = 1$ and $S_p(T) \subset S_2(T)$ [Halberg, Proc. Amer. Math. Soc. 8 (1957), 728-732; MR 19, 566; where the matrix T is not assumed to be a Toeplitz matrix]. If $I(T)$ is not empty, $2 \in I(T)$, so that $t(\theta) \sim \sum t_n e^{2\pi i n \theta}$ is of class $L^2(0, 1)$. In fact, $t(\theta)$ is essentially bounded, $\|t(\theta)\|_\infty = \|T\|_2$, and $S_2(T) = R(t)$, where $R(t)$ is the set of λ such that $\text{meas } \{\theta: |t(\theta) - \lambda| < \varepsilon\} > 0$ for every $\varepsilon > 0$ [Toeplitz; cf. F. Riesz, Les systèmes d'équations linéaires, Gauthier-Villars, Paris, 1913; pp. 171-180]. The authors prove that $S_p(T) = S_2(T)$ if (A): $t(\theta \pm)$ exists for all θ and $p \in I^0(T)$, the interior of $I(T)$. The proof depends on a theorem of Stečkin [Dokl. Akad. Nauk. SSSR (N.S.) 71 (1950), 237-240; MR 11, 504]. The last part of the paper gives results on the point spectrum $P_p(T)$, for ex-

ample: Let $E_\lambda = \bigcap_{\varepsilon>0} \{\theta: |\theta(\theta) - \lambda| > \varepsilon\}$, (A) hold, $x \in I_p$ and $(T - \lambda I)x = 0$; then $\sum x_n e^{2\pi i n \theta}$ is convergent with the sum 0 for $x \in E_\lambda$. As a type of converse: Let $t(\theta)$ be of bounded β variation (i.e., let $(\sum |\theta_{k+1} - \theta_k|)^\beta \leq \text{Const.}$ for all meshes on $[0, 1]$), $\phi \in I(T)$, $2 < \phi < \infty$, $x \in I_p$, $\sum x_n e^{2\pi i n \theta}$ convergent to 0 for $\theta \in E_\lambda$, m an integer and $m \geq \beta(\phi - 2)/2\phi$; then $(T - \lambda I)^m x = 0$. The proofs of these two theorems depend on Lebesgue summability of trigonometric series.
P. Hartman (Baltimore, Md.)

6006:

Keogh, F. R. Some theorems on Fourier series with random signs. J. London Math. Soc. 33 (1958), 284-288.
Let C be a class of integrable functions with Fourier series

$$(1) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and $E(C)$ be a set of t with the property that, for every series (1) such that the series

$$(2) \quad \frac{1}{2}a_0 r_0(t) + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r_n(t)$$

is the Fourier series of a function of C for all t in $E(C)$, the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \pm (a_n \cos nx + b_n \sin nx)$$

is also for all sets of signs.

It is known that every set of positive measure is an $E(S)$ and an $E(L^r)$ for $1 \leq r \leq 2$, but not necessarily either an $E(L^r)$ for $r > 2$ or an $E(B)$ (B being the class of essentially bounded functions and S the class of Fourier-Stieltjes series). The author proves that every set of the second category is an $E(S)$, an $E(L^r)$ for $r \geq 1$ and an $E(B)$. As a corollary, he proves that if the series (2) is the Fourier series of a function of B for a set of t of the second category, then $\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$. S. Izumi (Sapporo)

6007:

Flett, T. M. On the absolute summability of a Fourier series and its conjugate series. Proc. London Math. Soc. (3) 8 (1958), 258-311.

Si $f(\theta)$ est 2π -périodique et intégrable dans $(-\pi, \pi)$ on pose $\varphi(t) = f(\theta + t) - f(\theta - t)$, et, pour $\alpha > 0$, $t \geq 0$, désignons par $\Psi_\alpha(t)$ l'intégrale d'ordre α de Riemann-Liouville de φ , c'est-à-dire:

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u) du.$$

Soit

$$f(\theta) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

$$B_n(\theta) = a_n \sin n\theta - b_n \cos n\theta;$$

soit, enfin, I_n^β la n -ième moyenne de Cesàro d'ordre β de la série $\sum B_n(\theta)$. L'auteur démontre que, si $r \geq k > 1$, $\alpha \geq 0$, $\beta > \alpha + \sup(1/k, 1/r)$ ($1/r + 1/r' = 1$), on a:

$$\left\{ \sum_{n=1}^{\infty} n^{-1} |I_n^\beta| r \right\}^k \leq B \left\{ \int_0^\pi |\Psi_\alpha|^{k(1-\alpha)} dt \right\}^{1/k} + B \int_0^\pi |\varphi(t)| dt \quad (B = B(r, k, \alpha, \beta)).$$

S. Mandelbrojt (Paris)

6008:

Kumari, Sulaxana. Determination of the jump of a function by its Fourier series. Proc. Nat. Inst. Sci. India. Part A 24 (1958), 204-216.

Let

$$f(t) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt,$$

$$\theta(t) = \frac{1}{2}\{f(x+t) - f(x-t)\} - m, \theta_\alpha(t) = \alpha t^{-\alpha} \int_0^t (t-u)^{\alpha-1} \theta(u) du.$$

Also let $\tau_\beta(w)$ denote the (R, w, β) mean of the sequence $\{n(b_n \cos nx - a_n \sin nx)\}$. It is shown that if

$$(*) \quad \int_0^t |\theta_\alpha(u)| du = o(t^{\rho+1} / \log(1/t)) \text{ as } t \rightarrow 0,$$

or

$$(**) \quad \int_t^\pi |\theta_\alpha(u)| / u du = o(t^\rho (\log(1/t))^{\rho+1}), \rho > -1, \text{ as } t \rightarrow 0,$$

then

$$\tau_{\alpha+\rho+1} - 2m/\pi = o(w^{-\rho}) \text{ or } o(w^{-\rho} (\log(1/t))^{\rho+1}),$$

respectively, for $\alpha > 0$, $-1 < \rho < 1$, and $\alpha + \rho > 0$. The case $\rho = 0$ of (*) is related to the work of Mohanty and Nanda [Proc. Amer. Math. Soc. 5 (1954), 79-84; MR 15, 523].
P. Civin (Eugene, Ore.)

6009:

Artemiadis, Nicolas. Sur les transformées de Fourier et leurs applications aux séries et sur les fonctions typiquement réelles d'ordre ϕ . Ann. Sci. Ecole Norm. Sup. (3) 74 (1957), 269-318.

L'idée essentielle de ce travail est la suivante: f appartenant à L_1 , il s'agit d'obtenir des estimations de $|f|$, en faisant des hypothèses sur sa transformée de Fourier. Les théorèmes correspondants généralisent, en quelque sorte, les théorèmes connus portant sur les estimations des $|a_n|$ lorsque $\sum a_n z^n$ est, par exemple, une fonction univalente, ou une fonction typiquement réelle. D'ailleurs, ces théorèmes portant sur les fonctions de L_1 permettent de retrouver, tout en les améliorant, les théorèmes sur les séries de Taylor. Voici un exemple typique des résultats obtenus: Si $f(x)$ est continue sur R et $f \in L_1$; si $\text{Re } f \geq 0$ (partie réelle de la transformée de Fourier), $\text{Re } f(0) > 0$, $f \in L_1$, alors

$$|f(x)| + |f(-x)| \cos\{\arg f(x) + \arg f(-x)\} \leq 2 \text{Re } f(0), x \in R.$$

Voici aussi une application de ce théorème: si $F(z) = \sum_{n=0}^{\infty} a_n z^n$ est une fonction typiquement réelle, et si $\lim_{r \rightarrow 1-0} (1-r)^2 F(r) = 8l$, on a pour tout $r \geq 2$:

$$r^2 \leq 1 + a_2 + \dots + a_{r-1} + \frac{a_r}{2} \leq 2F(1)$$

($F(1)$ est, par définition, la limite de $F(r)$ lorsque $r \rightarrow 1-0$; cette limite, qui peut être infinie, existe toujours). Ce résultat améliore un résultat de Mandelbrojt, lorsque $F(1) = \infty$ [Mandelbrojt, Bull. Sci. Math. 58 (1934), 185-200]. Plusieurs autres résultats sont obtenus.

S. Mandelbrojt (Paris)

INTEGRAL TRANSFORMS

See also 5859, 5910, 6044.

6010:

Talalyan, A. A. Integral representation of measurable functions with kernels generating unitary transformations of the space $L_2(0, \infty)$. Akad. Nauk Armyan. SSR. Dokl. 26 (1958), 257-261. (Russian. Armenian summary)

Suppose U , and hence U^{-1} , are unitary operators of

$L_2(0, \infty)$ onto itself and are represented by the kernels $K(x, t)$ and $L(x, t)$, respectively, in the sense that

$$Uf(x) = \text{l.i.m.}_{a \rightarrow \infty} \int_0^a K(x, t)f(t)dt.$$

Suppose further that K and L are of Hilbert-Schmidt type on any bounded interval.

Theorem: For any measurable function f defined on $[0, \infty)$ there exists a measurable function τ , square integrable on any finite interval, so that $\int_0^a L(x, t)\tau(t)dt$ converges in measure to f on any finite interval in $[0, \infty)$ as $a \rightarrow \infty$.

A sequence $\{f_n\}$ of functions, finite a.e., is said to converge in the mean to f on the finite interval $[a, b]$ in the generalized sense if for every $\varepsilon > 0$ there exists a measurable set $E \subseteq [a, b]$ with $|E| > b - a - \varepsilon$ such that $\{f_n\}$ converges in the $L_2(E)$ mean to f .

For a function f which is finite a.e. the conclusion to the above theorem is the following: $\int_0^a L(x, t)\tau(t)dt$ converges to f in the mean in the generalized sense on each finite interval. In the case where $f=0$, the function τ may always be chosen different from the null function.

The proofs of these theorems require results previously obtained by the author [Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk. 10 (1957), no. 3, 17-34; MR 19, 742].

A. Devinatz (St. Louis, Mo.)

6011:

Carstoiu, John. La transformation de Stieltjes et le calcul symbolique. C. R. Acad. Sci. Paris 247 (1958), 1544-1546.

It is well known that the Stieltjes transform is an iterated Laplace transform. The author expresses this in the notation of symbolic (Heaviside) calculus and writes down the corresponding formula for the generalized Stieltjes transform $\int_0^\infty f(x)(p+x)^{-n-1}dx$ of $f(x)$. He also notes some other operational identities.

A. Erdelyi (Pasadena, Calif.)

6012:

Wuyts, P. Region of convergence of an integral of the form

$$\int_0^{+\infty} e^{-\lambda(t)} F(t) dt, (b.f.)$$

($\lambda(t)$ complex). Meded. Kon. Vlaamse Acad. Kl. Wetensch. 18 (1956), no. 3, 70 pp. (Dutch. English introduction)

Let $\lambda(t) = \alpha(t) + i\beta(t)$, where $\alpha(t)$ is monotone increasing and tends to infinity with t . Let $\lambda(t)$ have only discontinuities of the first kind, finite in number in any finite interval. Let

$$r = \liminf_{t \rightarrow \infty} \frac{\beta(t)}{\alpha(t)}, \quad R = \limsup_{t \rightarrow \infty} \frac{\beta(t)}{\alpha(t)},$$

let $\rho(y) = ry$, $y < 0$, $\rho(y) = Ry$, $y > 0$, and let $H(s_0)$ and $H^*(s_0)$ be the sectors $x - x_0 > \rho(y - y_0)$ and $x_0 - x > \rho(y_0 - y)$, respectively, where $s = x + iy$, $s_0 = x_0 + iy_0$. The author shows the existence of a quantity $P \geq 0$ with the property that if the integral

$$\int_0^\infty e^{-\lambda(t)} F(t) dt$$

converges for $s = s_1$ [diverges for $s = s_2$], then it converges for $s \in H(s_1 + P)$ [diverges for $s \in H^*(s_2)$]. Here P depends only on $\lambda(t)$. If $P = 0$, the region of convergence of the integral is simply-connected and there exists a function $\varphi(y)$ such that the integral converges for $x > \varphi(y)$ and diverges for $x < \varphi(y)$. If r and R are both finite, then $\varphi(y)$ is finite and continuous for $-\infty < y < +\infty$; if one of them is finite there exists an interval (y_1, y_2) in which $\varphi(y)$ has

only discontinuities of the first kind and outside of which $\varphi(y)$ is infinite. If $P > 0$ there may be isolated points of convergence, and it is shown by examples that the distance of such a point from the interior of the region of convergence may reach P . The greater part of the paper is devoted to the determination of $\varphi(y)$ for which a number of, necessarily complicated, formulas are given. A sample is given by

$$\varphi(y) = \limsup_{t \rightarrow \infty} \left\{ \frac{1}{\alpha(t)} \log \left| e^{-\alpha^2(t)} \int_0^t e^{\alpha^2(u) - t\gamma\lambda(u)} F(u) du \right| \right\}.$$

Here $\alpha^2(t)$ may be replaced by other functions satisfying certain conditions.

E. Hille (New Haven, Conn.)

6013:

Malyuzhinec, G. D. Inversion formula for Sommerfeld integral. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 1099-1102. (Russian)

The author shows that one solution to an integral transform

$$(*) \quad F(r) = (2\pi i)^{-1} \int e^{mr \cos \alpha} f(\alpha) d\alpha$$

is

$$f(\alpha) = (-\frac{1}{2})m \sin \alpha \int_0^\infty F(r) e^{-mr \cos \alpha} dr,$$

the integral in (*) being taken over a certain two-piece polygonal contour in the complex plane. This is the only solution satisfying an appropriate order condition at infinity. The result is used to solve the two-dimensional wave equation with new boundary conditions in a wedge-shaped domain.

R. R. Goldberg (Evanston, Ill.)

6014:

Džrbašyan, M. M.; and Nersesyan, A. B. On the use of some integrodifferential operators. Dokl. Akad. Nauk SSSR 121 (1958), 210-213. (Russian)

This note discusses relations between Dirichlet series and quasi-analytic functions related to the integrodifferential operators defining various kinds of fractional integration. For example, theorem 4 is analogous to Carleman's theorem which gives a sufficient condition for uniqueness of the moment problem.

M. M. Day (Urbana, Ill.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 5929, 6039.

6015:

Mysovskih, I. P. Representation of the resolvent of the sum of two kernels. Mat. Sb. N. S. 46(88) (1958), 77-90. (Russian)

For the Fredholm equation

$$(1) \quad \varphi(s) = \lambda \int_a^b K(s, t) \varphi(t) dt + f(s),$$

consider the related equation

$$(2) \quad \bar{\varphi}(s) = \lambda \int_a^b M(s, t) \bar{\varphi}(t) dt + f(s),$$

where $M(s, t)$ is an approximating kernel for $K(s, t)$, and set $N(s, t) = K(s, t) - M(s, t)$. Assuming K , M and $f(x)$ continuous on the basic square and interval, the results of the paper may be summarized as follows: (A) Fred-

holm's first minor for K , $D_K(s, \lambda)$, is expressed in terms of the first minors for M , N and their Fredholm determinants $D_M(\lambda)$, $D_N(\lambda)$ and minors of higher order; (B) using a formulation of Lalesco [Bull. Sci. Math. 42 (1918), 195-199] for $D_K(\lambda)$ in terms of $D_M(\lambda)$, $D_N(\lambda)$ and higher order minors, the result in (A) is manipulated into a fairly compact formula expressing the resolvent kernel for K , $R_K(s, t, \lambda)$ in terms of $R_M(s, t, \lambda)$, $R_N(s, t, \lambda)$ and terms depending on them; (C) the result in (B) is then used to derive an estimate for $|\varphi(s) - \bar{\varphi}(s)|$. It is assumed that λ is not a characteristic value for either $K(s, t)$ or $M(s, t)$. The formulas involved are much too complicated for inclusion in a brief review.

J. F. Heyda (Cincinnati, Ohio)

6016:

Suharevskii, I. V. On the stability of the solutions of integral equations in the case of a discontinuous variation of the kernel. Dokl. Akad. Nauk SSSR 122 (1958), 774-777. (Russian)

Consider the Fredholm equation

$$(1) \quad u(x) - \mu \int_0^1 K(x, s) u(s) ds = f(x),$$

where $\mu=1$ is a characteristic value, $f(x)$ is orthogonal over $(0, 1)$ to the corresponding characteristic functions of the associated kernel $K(x, s)$, and let M denote the set of solutions of (1) when $\mu=1$. The author replaces $K(x, s)$ by the discontinuous kernel $K_\lambda(x, s)$, which coincides with $K(x, s)$ for $\lambda < s < 1$ and is identically zero for $0 < s < \lambda$. With suitable restrictions on $K(x, s)$ and $f(x)$ he then states (without proof) theorems which insure, for sufficiently small λ , that (a) the equation

$$(2) \quad u_\lambda(x) - \int_0^1 K_\lambda(x, s) u_\lambda(s) ds = f(x)$$

will have a unique solution $u_\lambda(x)$; (b) $\lim_{\lambda \rightarrow 0} u_\lambda(x) = u_0(x) \in M$; (c) $u_0(x)$ will satisfy conditions which identify it in M .

J. F. Heyda (Cincinnati, Ohio)

6017:

Schmeidler, Werner. Variationsrechnung und Integralgleichungen. J. Reine Angew. Math. 200 (1958), 182-189.

A method frequently applied in the theory of (linear or non-linear) integral equations is as follows: If y is the unknown function one tries to construct a functional $\phi_0(y)$ such that the integral equation to be solved can be written in the form

$$(1) \quad E(y) = 0$$

where E is the "Euler operator" belonging to $\phi_0(y)$. The equation (1) is then a necessary condition for y to make $\phi_0(y)$ an extremum, so that the existence of an extreme value of $\phi_0(y)$ implies the existence of a solution of (1). This method (combined with an approximation procedure) was, e.g., used by A. Hammerstein in his classical paper on non-linear integral equations [Acta Math. 54 (1930), 117-176].

A quite similar method is followed in the present paper. For real valued functions $y=y(s)$ ($0 \leq s \leq 1$) the author considers the functional

$$\phi_0(y) = k \int_0^1 y^{n+1}(s) ds + \phi(y),$$

where k is a "large enough" constant and n an odd integer,

and where ϕ is of the form

$$\phi(y) = \sum \int_0^1 \cdots \int_0^1 K_{\alpha_1, \dots, \alpha_v}(s, t_1, \dots, t_v) \times y(s)^{\alpha_1} y(t_1)^{\alpha_1} \cdots y(t_v)^{\alpha_v} ds dt_1 \cdots dt_v + \int_0^1 f(s) y(s) ds;$$

the summation is extended over all integers $\alpha, \alpha_1, \dots, \alpha_v$ subject to the restrictions $n \geq \alpha > 0$, $n \geq \alpha_i \geq 0$, $i=1, 2, \dots, v$ and $2 \leq \alpha + \alpha_1 + \dots + \alpha_v \leq n+1$. The "Euler operator" $E(y)$ of $\phi_0(y)$ is then a polynomial of the form $E(y) = \sum_{i=0}^n y(s)^i a_i$, where the coefficients $a_i = a_i(s, y)$ are functionals of y depending on s . Due to the algebraic character of (1) complications may arise in the application of the method described above (multiple valuedness, ramifications). In order to avoid these complications the author puts restrictive conditions (for all s and y) on the polynomial $\sum_{i=0}^n i a_i$.

The author treats with the same method the eigenvalue problem of the integral equation (1) which arises if $\phi_0(y)$ is to be made an extremum under a side condition. For the definition of "eigenvalue" for the non-linear operator $E(y)$ we refer to an earlier paper of the author [Ann. Acad. Sci. Fenn. Ser. A. I, no. 220 (1955); MR 18, 302].

E. H. Rothe (Ann Arbor, Mich.)

6018:

Maurin, K. Elliptizität und schwache Halbstetigkeit gewisser Funktionale der Variationsrechnung mehrfacher Integrale. Vollstetigkeit Greenscher Transformationen. Studia Math. 17 (1958), 175-187.

Let H be a Hilbert space, and let (v, u) denote the scalar product of the elements v and u of H . A linear operator P on H is called positive definite if there exists a positive constant c such that $(Pu, u) \geq c(u, u)$. A linear (symmetric) operator A is called elliptic if $A = P + K$, where P is positive definite and K completely continuous. A (not necessarily quadratic) functional $I(u)$ is called elliptic if it is the sum of a positive definite quadratic form and a weakly continuous functional. The Hilbert spaces occurring in the paper are derived by extension from certain spaces of functions $u(x)$ defined for $x = (x_1, x_2, \dots, x_n)$ in a bounded domain Ω_n of the real Euclidean n -space, and the functionals $I(u)$ treated are sums of integrals extended over Ω_n or manifolds S_j of dimension $j < n$ situated in the closure $\bar{\Omega}_n$ of Ω_n , while the integrands are functions of x, u and partial derivatives

$$D^{\mu} u(x) = \frac{\partial^{\mu} u(x)}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}}, \quad |\mu| = \mu_1 + \cdots + \mu_n.$$

In the first section of this paper the author surveys shortly but clearly those results and methods of the modern theory of partial differential equations which are needed later on. In the second section he proves the ellipticity of the "Ehring quadratic functionals" [Math. Scand. 2 (1954), 267-285; MR 16, 706] which contain as a special case certain quadratic functionals used by Friedrichs [Math. Ann. 98 (1928), 206-247]. The third section deals with certain functionals $I(u)$ which are not necessarily quadratic. It is shown that the second differential $D^2(x, h, h)$ of the $I(u)$ treated is an elliptic quadratic functional of h . This fact implies the lower semicontinuity of I . In the fourth section the author returns to the functionals treated in the second section and gives necessary and sufficient conditions for the existence of a (unique) minimum. Section 5 deals with a "Green's transformation G_i " which is, essentially, the inverse of the elliptic differential operator (augmented by a term containing the parameter

δ) for which the quadratic functional treated is the "Dirichlet integral" [cf., e.g., the paper of Ehrling quoted above]. A simple proof for the complete continuity of G_δ (for large enough δ) is given, and various statements concerning the spectrum of G_δ are proved. In the last section it is pointed out that the spectral statements of section 5 are also true for the second differential $D^2(x, h, h)$ of the functional $I(u)$ discussed in section 3.

E. H. Rothe (Ann Arbor, Mich.)

6019:

Abian, Smbat; and Barnett, I. A. Functional invariants of a linear homogeneous integro-differential equation. *Duke Math. J.* 25 (1958), 547-552.

The authors use the semi-canonical form of the integro-differential equation

$$y^{(m)}(x; t) + \int_0^1 a_1(x, u; t) y^{(m-1)}(u; t) du + \dots + \int_0^1 a_m(x, u; t) y(u; t) du = 0$$

(where differentiations are with respect to t) to find functional invariants of this equation under an invertible Fredholm transformation

$$y(x; t) = \bar{y}(x; t) + \int_0^1 k(x, u; t) \bar{y}(u; t) du.$$

A. F. Ruston (Sheffield)

FUNCTIONAL ANALYSIS

See also 5782, 5808, 5818, 5820, 5880, 5967, 6000, 6001, 6002, 6005.

6020:

Fullerton, R. E. An intersection property for cones in a linear space. *Proc. Amer. Math. Soc.* 9 (1958), 558-561.

Let X be a real linear space. A subset A of X is linearly closed if it intersects every line in a closed subset of the line. A non-empty subset C of X is a cone with vertex θ if, whenever $x \in C$, $\lambda x \in C$ for every $\lambda \geq 0$. If C is a convex cone with vertex θ and $z \in (x+C) \cap (y+C)$, then clearly $z+CC(x+C) \cap (y+C)$. If C is a linearly closed convex cone with vertex θ , K is a cone with vertex θ and $z+KCx+C$, then KCC (for, if $k \in K$, then $z+nk \in x+C$ and so $k+(z-x)/n \in C$ for every positive n). It follows that, if $(x+C) \cap (y+C) = z+K$, then $K=C$. This is the principal result of the paper under review. The author then considers circumstances in which this sort of situation arises.

A. F. Ruston (Sheffield)

6021:

Bessaga, C.; Pełczyński, A.; and Rolewicz, S. Some properties of the norm in F -spaces. *Studia Math.* 16 (1957), 183-192.

Let X be an F^* space (F space which is not necessarily complete) with metric d . The authors prove that there exists a metric d_1 equivalent to d such that for all $x \in X$ and $t > 0$, $d_1(tx, \theta)$ is a strictly monotone, concave and infinitely differentiable function of t . Furthermore, given an F^* space, there exists a metric such that $\sup_x d(x, \theta) = +\infty$ if and only if there exists a neighborhood U such that, for any positive integer n , $U^n = U \oplus \dots \oplus U \subset X$. Also, a characterization is given of those F spaces such that there exists a sequence a_n such that $\lim a_n = +\infty$ and $\limsup a_n d(x/n, \theta) < +\infty$ for every $x \in X$.

L. Brown (Detroit, Mich.)

6022:

Morse, Marston; and Transue, William. Vector subspaces A of C^B with duals of integral type. *J. Math. Pures Appl.* (9) 37 (1958), 343-363.

This paper develops ideas set forth in an earlier work by the same co-authors [*J. Analyse Math.* 4 (1954/55), 149-186; MR 17, 469]. Apart from minor typographical variations the notations are as before. The paper deals with MT -spaces, i.e., vector subspaces A of C^B equipped with a non-trivial seminorm \mathcal{N}^A , including \mathcal{X}_C as a dense subspace, and such that $|x|$ and \bar{x} belong to A whenever $x \in A$. If A' is the dual of A , $\alpha \rightarrow \hat{\alpha} = \alpha|_{\mathcal{X}_C}$ defines a homomorphism of A' onto a subspace \mathcal{A}' of \mathcal{X}_C (complex Radon measures on E). It is supposed that each $x \in A$ is integrable for each $\hat{\alpha} \in \mathcal{A}'$ and that $\alpha(x) = \int x d\hat{\alpha}$ ($x \in A$, $\alpha \in A'$); this implies that $\alpha \rightarrow \hat{\alpha}$ is an isomorphism. (Whilst a notational distinction is preserved without exception between A' and its isomorph \mathcal{A}' , that between α and $\hat{\alpha}$ appears to be waived at some points. In this review both identifications are made. There seems also to be some confusion over the symbols \mathcal{L}_C and \mathcal{L}_C^1 , independent definition of the former being lacking.) The principal result (Theorem 7.2) gives conditions under which an MT -space A coincides with the generally larger derivative space $\Omega^A = \bigcap_{\alpha \in A'} \mathcal{L}_C^1(\alpha)$. Two properties of A are required for this to hold. The first is that A be maximal, i.e., be complete with respect to the extension of \mathcal{N}^A defined for all $x \in C^B$ by

$$\mathcal{N}^A(x) = \text{Sup} \left\{ \int^* |x| d|\alpha| : \alpha \in A', \|\alpha\|_{A'} \leq 1 \right\},$$

where

$$\|\alpha\|_{A'} = \text{Sup} \{ \alpha(x) : x \in A, \mathcal{N}^A(x) \leq 1 \}.$$

The second requirement is that A be of "Cauchy type", the content of which is as follows. Let B be the real MT -space $A \cap R^B$ and form the quotient \mathbf{B} of B modulo the kernel of \mathcal{N}^B (extended as above). A is said to be of Cauchy type if it is a maximal MT -space such that each norm-bounded subset H of \mathbf{B} which is filtering for \leq defines a Cauchy filter on \mathbf{B} . Numerous subsidiary results are also established.

R. E. Edwards (Woking)

6023:

Goffman, Casper. Completeness in topological vector lattices. *Amer. Math. Monthly* 66 (1959), 87-92.

As an application of a theorem (monotone completeness implies topological completeness) in a paper by H. Nakano [*J. Fac. Sci. Hokkaido Univ. Ser. I* 12 (1953), 87-104; MR 15, 137], the author proved that Köthe spaces, defined here more generally than in a paper by J. Dieudonné [*J. Analyse Math.* 1 (1951), 81-115; MR 12, 834], are topologically complete for some topologies.

H. Nakano (Sapporo)

6024:

Cristescu, Romulus. Théorème de Radon-Nikodym dans les K -espaces. *An. Univ. "C. I. Parhon" București. Ser. Ști. Nat.* 6 (1957), no. 14, 25-27. (Romanian. French and Russian summaries)

Let X be a K space (complete vector lattice). It has been shown [Kantorovič, Vulih and Pinsker, *Functional analysis in partially ordered spaces*, Moscow, 1950; MR 12, 340] that there exists a complete Boolean algebra X^* such that X is, in a certain sense, representable as a sub-direct sum of elements of X^* . For any $e \in X^*$ let $(e)x$ be the e component of x in this representation. The following theorem is proved.

Let φ be real valued, additive, bounded and order continuous on X^* . Let f be an order continuous, linear, positive functional with the following property: if $H \subset X$ is a directed set such that $x \in H$, $0 \leq x' \leq x$ implies $x' \in H$, and if $\sup_{x \in H} f(x) < \infty$, then there exists a bounded sequence $\{x_n\} \in H$ such that $\sup_{x \in H} f(x) = \lim_n f(x_n)$. Conclusion: if, for $e \in X^*$, $f(e) = 0$ implies $\varphi(e) = 0$, there exists an element $x_0 \in X$ such that $\varphi(e) = f(e)x_0$.

R. E. Fullerton (College Park, Md.)

6025:

Gurevič, L. A. On equivalent systems. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 2 (1956), 47-54. (Russian)

Two systems $\{e_i\}$ and $\{e_i^*\}$ of elements of a Banach space B are called equivalent if there exists an isomorphic mapping of the space taking one system into the other. It is proved that the properties of a system of elements of being complete, minimal, a basis, etc., are preserved under isomorphic mappings of the space. Two criteria are given for equivalence of complete systems.

The second part of the article is devoted to the conditions for equivalence of displacements. A displacement of a system $\{e_i\}$ is defined as a system $\{e_i^*\}$ determined by the equations

$$e_i^* = e_i - \lambda^{-1} h_i \quad (i = 1, 2, \dots),$$

where $\{h_i\}$ is a given system of elements and λ is a given number. By making use of a general theorem which he proved about the equivalence of displacements, the author obtains a series of conditions for equivalence of displacements in the case of concrete systems. From them follow certain already known theorems for the case that the system $\{e_i\}$ is a basis [N. K. Bari, Dokl. Akad. Nauk SSSR 54 (1946), 379-382; MR 8, 513; K. I. Babenko, same Dokl. 57 (1947), 427-430; MR 9, 142; M. Krein, D. Milman, and M. Rutman, Zapiski Inst. Mat. Mech. Harkov 16 (1940), 106-110; MR 3, 49].

Ya. B. Rutickii (RZMat 1958 #3888)

6026:

Efimov, N. V.; and Stečkin, S. B. Čebyšev sets in Banach spaces. Dokl. Akad. Nauk SSSR 121 (1958), 582-585. (Russian)

The authors call a subset M of a normed linear space X a Čebyšev set if to each point of X there corresponds a unique nearest point in M . The theorem which states that (for certain spaces X) Čebyšev sets are convex has appeared sporadically since 1935 [Th. Motzkin, Atti Acad. Naz. Lincei Rend. (6) 21 (1935), 562-567], the usual assumptions on X being smoothness of the unit cell and finite dimensionality. The authors extend this theorem, for compact M , to infinite dimensional spaces whose unit cells are smooth and which have "uniformly small curvature", a somewhat complicated property possessed by Hilbert space but not by all uniformly convex spaces. {The result for compact Čebyšev subsets of Hilbert space follows from a general theorem of V. L. Klee [Trans. Amer. Math. Soc. 74 (1953), 10-43; MR 14, 989; Theorem A 2.5].}

R. R. Phelps (Princeton, N.J.)

6027:

Phelps, R. R. Subreflexive normed linear spaces. Arch. Math. 8 (1957), 444-450.

A normed linear space E is called subreflexive if the set P of all linear functionals in E^* which attain their maximum on S , the unit sphere of E , is norm-dense in E^* . It is well known that if E is reflexive then $P = E^*$. For separable Banach spaces (=complete linear normed

spaces) the converse of this statement has recently been proved by R. C. James [Ann. of Math. (2) 66 (1957), 159-169; MR 19, 755]. But for non-separable spaces this is still an open question. This seems to justify the study of subreflexive spaces. The following characterization of subreflexivity is given: A normed linear space is subreflexive if and only if each bounded closed convex set C is orthodox (i.e., $x \notin C$ implies there exists $f \in P$ such that $f(x) > \sup f(C)$). As remarked at the end of the paper, the proof that this condition is sufficient is not correct, but a correct version will be published in a subsequent volume of this journal [9 (1958), 439-440; MR 21 #2175]. Some sufficient conditions for subreflexivity are given such as: E is subreflexive if its conjugate E^* is locally uniformly convex (i.e., $\|f\| = 1 = \|f_n\|$ and $\|f + f_n\| \rightarrow 2 \Rightarrow \|f - f_n\| \rightarrow 0$). An example shows that there are non-reflexive Banach spaces which have locally uniformly convex conjugates. By means of the following theorem of V. L. Klee: Every linear topological vector space E which has a neighborhood basis at the origin with cardinal \leq the dimension of E has a dense Hamel basis—an example is constructed of a linear normed space which has a reflexive completion which is isomorphic to a subreflexive space, but which is not subreflexive. Then the Banach spaces c_0 , l_1 , m , $L_1(\mu)$ (where μ is a totally σ -finite measure), $C(X)$ (where X is a compact Hausdorff space) are shown to be subreflexive. It seems possible that every Banach space is subreflexive.

W. A. J. Luxemburg (Pasadena, Calif.)

6028:

Nikol'skii, S. M. An imbedding theorem for functions with partial derivatives considered in different metrics. Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 321-336. (Russian)

What happens if, in the definition of the Nikol'skii class $H_p^{(r_1, \dots, r_n)}$ for functions of n variables, one allows different L_p norms for the variations involving each of the n variables? In particular, do the earlier results generally carry over to these new classes $H_{p_1, \dots, p_n}^{(r_1, \dots, r_n)}$? This question, raised in the case of the embedding theorem by S. L. Sobolev, is here answered in the affirmative.

M. G. Arsove (Seattle, Wash.)

6029:

Gál, I. S. On the foundations of the theory of distributions. Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 1248-1252.

In this paper the author summarizes his results on Schwartz distributions considered as the elements of the completion of a suitable uniform structure for the linear space of those real-valued functions on a finite dimensional Euclidean space S which have compact supports and possess mixed derivatives of all orders on S . For this purpose the author introduces the notion of a distribution structure for an inner product space X . Let a family of sets U be given on X ; then the distribution structure is the uniform structure associated with the topology of uniform convergence on the sets U . If the U 's are finite sets, then he calls this the uniform structure of weak distributions. In the case that X is the inner product space of the real-valued functions on a finite dimensional Euclidean space S which have compact supports and mixed derivatives of all orders, the U 's are sets of functions having a common compact support and $\|AU\| < +\infty$ for every $A \in \mathfrak{A}$, where \mathfrak{A} is the semi-group of all mixed derivatives $A: X \rightarrow X$. The main problem which is tackled in this paper is the problem of extending an operator $A: X \rightarrow X$ to the completion \bar{X} of X and inverting it. His results then lead to a generalization of a result of Schwartz

that every individual distribution can be divided by a regular function, namely, the author shows that the operator which consists of multiplying by a regular function has an inverse. *W. A. J. Luxemburg* (Pasadena, Calif.)

6030:

Love, E. R. A Banach space of distributions. II. *J. London Math. Soc.* 33 (1958), 288-306.

The investigations of part I [*J. London Math. Soc.* 32 (1957), 483-498; MR 19, 756], concerned with distributions which are $(k+1)$ order derivatives of summable functions, are continued, giving results on the differentiability of such distributions, on their products by differentiable functions and on the differentiability of such products. *J. L. B. Cooper* (Cardiff)

6031:

Bogolyubov, N. N.; and Vladimirov, V. S. On analytic continuation of generalized functions. *Izv. Akad. Nauk SSSR Ser. Mat.* 22 (1958), 15-48. (Russian)

For the study of dispersion relations in quantum electrodynamics, analytic extension of distributions in several variables is required. This has been carried out by L. Schwartz for one variable and by J. Lions [*J. Analyse Math.* 2 (1953), 369-380; MR 15, 307], L. Schwartz [*Medd. Lunds Univ. Mat. Sem. Tome Supplémentaire* (1952), 196-206; MR 14, 639; Prop. 8] for several variables. The authors here give an elaborate development of the topic with exact descriptions of the domains of analyticity obtained, for various tempered distributions of several variables. The results are too complicated to reproduce here. *E. Hewitt* (Seattle, Wash.)

6032:

Kuribayashi, Akikazu. On continuability of bilinear differentials. *Kōdai Math. Sem. Rep.* 10 (1958), 105-108.

Let D be a domain in the z -plane, and let $\varphi(z, t)$ be a Hermitian kernel, analytic in z and \bar{t} for $z, t \in D$. $\varphi(z, t)$ is said to be positive-definite if, for any z_1, z_2, \dots, z_n in D and arbitrary complex numbers c_1, c_2, \dots, c_n , we have

$$Q(\varphi) = \sum_{i,j=1}^n \varphi(z_i, z_j) c_i \bar{c}_j \geq 0.$$

If $\varphi(z, t)$ is another kernel of this type, the symbol $\varphi \ll \psi$ means that the quadratic form $Q(\psi) - Q(\varphi)$ is also positive-definite. In terms of these symbols, the following results are proved: (a) If $\varphi(z, t)$ is a positive-definite kernel defined in an open subset D_1 of D and

$$(*) \quad \varphi(z, t) \ll k(z, t) \text{ in } D_1,$$

where $k(z, t)$ is the Bergman kernel of D , then $\varphi(z, t)$ can be continued, and the relation $(*)$ holds, throughout D ; (b) if $\varphi(z, t)$ is a positive-definite kernel in D and belongs to the Lebesgue class $L^2(D)$ for fixed t , there exists a positive constant λ such that $\lambda \varphi(z, t) \ll k(z, t)$.

Z. Nehari (Pittsburgh, Pa.)

6033:

Esser, Martinus. A characterization of L^p spaces. *Portugal. Math.* 17 (1958), 19-39.

Let \mathfrak{B} be a Banach space. Under the assumption: \mathfrak{B} contains a set Γ for which a multiplication is defined such that for $\gamma_1, \gamma_2, \dots, \gamma_N \in \Gamma$, we have $\gamma_1 \gamma_2 \in \Gamma$; $\gamma_1 - \gamma_1 \gamma_2$ is the sum of finitely many elements in Γ ; $\gamma_1 \gamma_1 = \gamma_1$; $\gamma_1 \gamma_2 = \gamma_2 \gamma_1$; $(\gamma_1 \gamma_2) \gamma_3 = \gamma_1 (\gamma_2 \gamma_3)$; $\sum_{n=1}^N \gamma_n \in \Gamma$ implies $\gamma \sum_{n=1}^N \gamma_n = \sum_{n=1}^N \gamma \gamma_n$; and Γ is linearly dense in \mathfrak{B} (\mathfrak{B} is the smallest

closed linear subset containing Γ); the author proves that \mathfrak{B} is isometric to the L^p space over some measure space if there exist $1 \leq p < +\infty$ such that

$$\left\| \sum_{n=1}^N c_n \gamma_n \right\|^p = \sum_{n=1}^N \|c_n \gamma_n\|^p$$

for any orthogonal sum from Γ (c_1, c_2, \dots, c_N real numbers and $\gamma_1, \gamma_2, \dots, \gamma_N \in \Gamma$ such that $\gamma_n \gamma_m = 0$ for $n \neq m$). The author points out especially that this characterization of L^p space does not use partial ordering of \mathfrak{B} . However, the reviewer wants to remark that, defining $\sum_{n=1}^N c_n \gamma_n \geq \sum_{n=1}^N b_n \gamma_n$ for orthogonal sums from Γ as $c_n \geq b_n$ for all $n=1, 2, \dots, N$, we can make \mathfrak{B} a Banach lattice with

$$\|a \vee b\|^p \leq \|a\|^p + \|b\|^p \leq \|a+b\|^p \text{ for } 0 \leq a, b \in \mathfrak{B}.$$

Thus this characterization is reduced to that in a paper by H. Nakano [*Proc. Imp. Acad. Tokyo* 17 (1941), 301-307; MR 7, 249]. Furthermore, the idea of representation is the same as that in another paper [H. Nakano, *Proc. Phys.-Math. Soc. Japan* (3) 23 (1941), 485-511; MR 3, 210].

H. Nakano (Sapporo)

6034:

Schwartz, L. Généralisation des espaces L^p . *Publ. Inst. Statist. Univ. Paris* 6 (1957), 241-250.

Let X be a locally compact topological space which is a countable union of compact sets. A definition of the symbol $f(\mu)^{1/p}$, where μ is a positive measure on X and $|f|^p$ is integrable or locally integrable with respect to μ , is given as an equivalence class of pairs (f, μ) . By means of these $1/p$ -measures one may attach an L^p -space or $L^{loc p}$ -space intrinsically to a measure class. These spaces associated with various measure classes are all contained in one Banach space \mathfrak{S}^p or Fréchet space \mathfrak{S}_{loc}^p .

W. F. Stinespring (Princeton, N.J.)

6035:

Raikov, D. A. Complete continuity of an adjoint operator. *Dokl. Akad. Nauk SSSR (N.S.)* 119 (1958), 446-449. (Russian)

This note discusses locally convex linear topological spaces, X and Y , and their conjugates, X' and Y' , provided with the strong topology. A linear operator φ from X into Y is called completely continuous [or bounded] if there is in X a neighborhood of zero whose image in Y is compact [or bounded]. The author defines two classes of spaces, type (N) and type (M_0) , and shows that (theorem 2) if X is of type (M) , then for each bounded φ from X into Y , φ' , the conjugate of φ , is completely continuous from Y' to X' , and that (theorem 3) if Y is of type (N) , then for each bounded φ from X into Y , φ' is completely continuous. It is asserted that the converses hold. It is shown that type (N) includes all metrizable locally convex spaces and also includes inductive limits of sequences of spaces of type (N) .

M. M. Day (Urbana, Ill.)

6036:

Feldman, J. A remark concerning a theorem of B. Friedman. *Proc. Amer. Math. Soc.* 9 (1958), 551-552.

The theorem of Friedman [*Comm. Pure Appl. Math.* 8 (1955), 539-550; MR 17, 1229] states that if T is a densely defined linear operator with closed range in a Hilbert space H with a densely defined adjoint T^* having closed range, and if for $\varphi, \psi \in H$ the operator $\varphi \otimes \psi$ is defined by $\varphi \otimes \psi(x) = (x, \varphi)\psi$, then $T + \varphi \otimes \psi$ has closed range. The theorem proved here states that if T is a closed densely defined operator with closed range, then $T + \varphi \otimes \psi$ has closed range. *R. E. Fullerton* (College Park, Md.)

6037:

Rosenblum, M. On a theorem of Fuglede and Putnam. J. London Math. Soc. 33 (1958), 376-377.

Suppose b is a bounded operator acting on a Hilbert space H , and n is a normal (possibly unbounded) operator in H , with adjoint n^* . It was proved by B. Fuglede [Proc. Nat. Acad. Sci. 36 (1950), 35-40; MR 11, 371] that the relation $bnCnb$ implies bn^*Cn^*b . This was generalized by C. R. Putnam [Amer. J. Math. 73 (1951), 357-362; MR 12, 717] as follows: if m, n are normal and b is bounded, the relation $bnCmb$ implies bn^*Cm^*b .

In the paper under review, a concise proof is given for the Putnam result. The case that m, n are bounded is treated first, with an application of Liouville's theorem on the constancy of bounded entire functions. The general case is deduced from the bounded case by employing spectral projections of m, n .

In particular, suppose b, m, n are elements of an abstract C^* -algebra [see paragraph 11C of L. H. Loomis' *An introduction to abstract harmonic analysis*, D. van Nostrand, Toronto-New York-London, 1953; MR 14, 883]. If m, n are normal, then $bm=mb$ implies $bn^*=m^*b$. Until now, the proof of this result has depended on invoking the Gelfand-Neumark representation theorem for C^* -algebras. The proof under review is non-spatial, i.e., avoids the representation theorem.

S. K. Berberian (Iowa City, Iowa)

6038:

Rosenblum, Marvin. On the Hilbert matrix. II. Proc. Amer. Math. Soc. 9 (1958), 581-585.

[For part I see same Proc. 9 (1958), 137-140; MR 20# 1139.] The Hilbert matrix H_k has as its (m, n) th element $n+m+1-k$, where k is real and not a positive integer. It defines a bounded symmetric linear operator on l^2 . W. Magnus [Amer. J. Math. 72 (1950), 699-704; MR 12, 836] showed that the l^2 spectrum of H_0 is purely continuous and consists of $[0, \pi]$. The principal result of the present paper is that H_k has continuous spectrum of multiplicity 1 on $[0, \pi]$; if $k \leq \frac{1}{2}$, H_k has no point spectrum; if $k > \frac{1}{2}$, and p and q are the largest non-negative integers such that $2p < k - \frac{1}{2}$ and $2q < k - 3/2$, then $\pi \csc \pi k$ and $-\pi \csc \pi k$ are eigenvalues of H_k of multiplicities $p+1$ and $q+1$; H_k has no other point spectrum.

R. P. Boas, Jr. (Evanston, Ill.)

6039:

Butzer, Paul L. Zur Frage der Saturationsklassen singulärer Integraloperatoren. Math. Z. 70 (1958), 93-112.

Let B be a Banach space and let $T(\xi)$ ($0 < \xi < \infty$) be a uniformly bounded family of linear operations on B such that $\lim_{\xi \rightarrow 0+} T(\xi) = I$ in the strong operator topology. The author addresses himself to the following two questions. (i) Given any function $\omega(\xi) \downarrow 0$ as $\xi \downarrow 0$, for what elements $x \in B$ does $\|x - T(\xi)x\| = O(\omega(\xi))$ as $\xi \downarrow 0$. (ii) Does there exist a function $\omega_0(\xi)$ such that $\|x - T(\xi)x\| = o(\omega_0(\xi))$ as $\xi \downarrow 0$ implies $x - T(\xi)x = 0$, but such that the class C_0 of elements x for which $\|x - T(\xi)x\| = O(\omega_0(\xi))$ is not empty. These questions are examined in some dozen special cases. A typical example is as follows. Let $B = L^p(-\infty, \infty)$ ($1 < p < \infty$), and let

$$T(\xi)x(t) = (\pi\xi)^{-1} \int_{-\infty}^{\infty} x(t+u) e^{-u^2/\xi} du;$$

then $\omega_0(\xi) = \xi$ and C_0 is the set of x such that x and x' are totally continuous and $x'' \in L^p(-\infty, \infty)$. In many of the

examples considered the $T(\xi)$ form a semi-group and the theory of semi-groups can be applied to study the questions under consideration.

I. I. Hirschman, Jr. (St. Louis, Mo.)

6040:

Schreiber, Morris. A functional calculus for general operators in Hilbert space. Trans. Amer. Math. Soc. 87 (1958), 108-118.

An operational calculus for general bounded operators on a Hilbert space is developed similar to that for normal operators in terms of spectral measures. An operator measure is a function F from Borel sets on the unit circle C in the complex plane to positive operators on a Hilbert space H which is weakly countably additive and such that $F(C) = I$. Then if f is a complex valued function bounded and Borel measurable on $\Lambda(F)$, the support of F on C ($f \in L_\infty(F)$), the weak integral $\int f(z) dF(z)$ defines a bounded linear operator T on H . It is known that to every linear operator A on H with $\|A\| \leq 1$ there corresponds uniquely an operator measure F such that $A = \int z dF(z)$ and $\int z^n dF(z) = (\int z dF(z))^n$ for all n [B. Sz. Nagy, Acta Sci. Math. Szeged 15 (1953), 87-92; MR 15, 326; M. Schreiber, Duke Math. J. 23 (1956), 579-594; MR 18, 748]. For $f \in L_\infty(F)$, let φ map $L_\infty(F)$ into the space of bounded operators on H by the mapping $f(A) = \int f(z) dF(z)$. The principal theorem establishes the fact that if F has the same null sets as Lebesgue measure on C , then φ is a norm decreasing algebraic homomorphism from the algebra of bounded boundary values of functions analytic inside the unit circle into the strong closure of the algebra generated by A and I . If the spectrum of A includes C then φ is an isometry of the space of continuous functions on C which are pointwise limits of a uniformly bounded sequence of polynomials onto the uniform closure of the algebra determined by A and I .

R. E. Fullerton (College Park, Md.)

6041:

Kužel', A. V. The reduction of unbounded non-selfadjoint operators to triangular form. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 868-871. (Russian)

Consider closed linear operators A in a Hilbert space; let D_A denote the domain of A , let G_A consist of $x \in D_A$ such that $(Ax, y) = (x, Ay)$ for all $y \in D_A$, and let A_0 denote the restriction of A to G_A . If D_A is dense, A_0 is Hermitian with deficiency index (r, r) , and $\dim D_A = r \pmod{G_A}$, where $r > 0$ may be infinite, then A is called quasi-Hermitian of rank r , or a K^r -operator. Various results of M. S. Livšic [same Dokl. 84 (1952), 873-876, 1131-1134; Mat. Sb. N.S. 34(76) (1954), 145-199; MR 14, 184, 185; 16, 48] on "triangular models" of bounded non-Hermitian operators are extended, with necessary modifications, to K^1 -operators, and some hints are given concerning analogous theorems for K^r -operators, $1 < r < \infty$.

M. Kašitov (Prague)

6042:

Hille, Einar. Problème de Cauchy: existence et unicité des solutions. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 1 (49) (1957), 141-143.

A résumé of the theory of the "Abstract Cauchy problem", inaugurated by the author and developed in various occasions, in particular, in sect. 23 of the author's book, written jointly with R. S. Phillips [Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., New York, N.Y., 1957; MR 19, 664].

K. Yosida (Tokyo)

6043:

Krein, S. G.; and Sobolevskii, P. E. A differential equation with an abstract elliptical operator in Hilbert space. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 233-236. (Russian)

Consider the initial value problem $dv/dt + Av = 0$, $v(0) = v_0$, where A is a linear operator in Hilbert space with a dense domain of definition $D(A)$, v_0 is an arbitrary element of $D(A)$, and $v = v(t)$ is desired to be a strongly continuous function with values in $D(A)$. This initial value problem for A is proper if (1) a solution exists for each v_0 in $D(A)$, (2) the solution for given v_0 is unique, and (3) the solution depends continuously on the initial value v_0 uniformly in t . Theorem 1: In order that the initial value problem for A be proper it is necessary that the closure \bar{A} of A be the infinitesimal generator of a strongly continuous semigroup $U(t)$ of bounded operators. If A is closed, the condition is also sufficient. The proof is said to follow the method used by Hille [Univ. e Politec. Torino. Rend. Sem. Mat. 12 (1953), 95-103; MR 15, 718]. For A positive definite on H , $U(t) = e^{-At}$ is the appropriate semigroup, and the solution of the initial value problem is $v(t) = U(t)v_0$.

B is called of fractional order with respect to the positive definite A if there are a $\gamma < 1$ and a K_γ such that $\|B\| \leq K_\gamma \|A^\gamma\|$, where $A^\gamma = \int_0^\infty t^{\gamma-1} e^{-At} dt$. An operator S is elliptic if there exists a positive definite A and a B of fractional order with respect to A such that $S = A + B$. Theorem 3. The initial value problem for an elliptic operator is proper.

Generalizations and applications are discussed briefly.

M. M. Day (Urbana, Ill.)

6044:

Hadwiger, H. Über die kontinuierliche Integrationsgruppe bei Ultrafunktionen. Arch. Math. 9 (1958), 211-218.

Let $\mathcal{E}(R)$ be the algebra of endomorphisms of a linear space R . Let $G = (-\infty, \infty)$ and $G^* = \{0, \pm 1, \pm 2, \pm 3, \dots\}$. It will be convenient to denote by $\mathcal{B}(G, R)$ the set of all homomorphisms of the additive group G into $\mathcal{E}(R)$. Of basic importance for this article are the properties of the Titchmarsh group $\{J_\alpha^*: \alpha \in G\}$ [see Krabbe, Proc. Amer. Math. Soc. 6 (1955), 219-225; MR 16, 1031]; it satisfies the condition $J^* \in \mathcal{B}(G, L^2(G^*))$, and J_{-1}^* is the translation operator defined for any α in $L^2(G^*)$ by the relation $[J_{-1}^* \alpha]_n = \alpha_{n+1}$ (when $n \in G^*$). The pair (R, J) is called a "vollständiges kontinuierliches Integrationssystem" (v.k.I.) if R is a linear space of differentiable functions and J is a member of $\mathcal{B}(G, R)$ such that $J_{-1} = J'$ and $J_0 = I$ whenever $f \in R$. If (R, J) is a v.k.I. and if $\alpha < 0$, then J_α is a differentiation operator of fractional order; however, the group property $J_\alpha J_\beta = J_{\alpha+\beta}$ considerably restricts the space R . In a previous article [Vierteljahr. Naturforsch. Ges. Zürich 92 (1947), 31-42; MR 8, 569], the author has embedded the space R_1 of entire functions into a cartesian product $R = R_1 \times R_2$ (with $R_2 \subset R_1$); thus $R_1 \subset R$ in the same sense that G is a subset of the complex field \mathbb{C} . Members of the linear space R are called "ultrafunctions". In the present article, R is made into a linear space of "differentiable" functions by the expedient of defining the "derivative" f' of any member $f = (f_1, f_2)$ of R by the relation $f' = (g_1, g_2)$, where

$$g_1 = f_1' \text{ and } g_2(z) = z^{-1} f_1(0) + \int_0^z f_2$$

(whenever $z \in \mathbb{C}$). Next, for any α in G , the operator J_α is the member of $\mathcal{E}(R)$ defined by a certain matrix (Q_{mn}^α)

($m, n = 1, 2$) of integral transformations; if $f = (f_1, f_2) \in R$, then $J_\alpha f = (g_1, g_2)$, where $g_n = \sum Q_{mn}^\alpha f_m$ ($n, m = 1, 2$). For example, Q_{11}^α is a member of $\mathcal{E}(R_1)$ defined as a composition of a Laplace transformation, a multiplication transformation, and a Fourier transformation. The property $J_\alpha J_\beta = J_{\alpha+\beta}$ is verified. The pair (R, J) is therefore a v.k.I., since R is a linear space of "differentiable" functions, since $J \in \mathcal{B}(G, R)$ and $J_{-1} = J'$ when $f \in R$. By defining on R a suitable norm, the relation $\|J_\alpha f\| = \|f\|$ holds whenever $f \in R$ and $\alpha \in G$; furthermore, the normed linear space $R^0 = \{f \in R: \|f\| < \infty\}$ is the isometric image of $L^2(G^*)$ under the one-to-one mapping $a \rightarrow f$, where

$$f_1(z) = \sum a_n z^n / n! \text{ and } f_2(z) = \sum a_{-k} z^k / k!$$

($v \in G^*, v \geq 0, k \in G^*, k > 0, z \in \mathbb{C}$). This mapping identifies the Titchmarsh operator J_α^* with the restriction to R^0 of the operator J_α . G. L. Krabbe (Lafayette, Ind.)

6045:

Sya, Do-Sin. Positive functionals of algebras. Dokl. Akad. Nauk SSSR 121 (1958), 233-235. (Russian)

Let A be a commutative algebra with unit over the real numbers and let \mathfrak{M} be its space of maximal ideals. x in A is non-negative if $M(x) \geq 0$ for every M in \mathfrak{M} ; a linear functional f on A is positive if f is not 0 and $f(x) \geq 0$ for every non-negative x in A . The author quotes the Moscow dissertation of Zarhina for an example of an algebra A and an f with $f(x^2) \geq 0$ for every x in A but f not positive. He then proves the following conditions on F and A equivalent if A has a countable set of generators. (1) F is positive. (2) There is a positive measure μ on \mathfrak{M} such that $F(x) = \int_{\mathfrak{M}} x(M) d\mu(M)$ for all x in A . (3) For each real non-negative polynomial $p(t_1, \dots, t_n)$ of degree ≤ 4 it follows that $F(p(x_1, \dots, x_n)) \geq 0$ for all x_1, \dots, x_n in A . M. M. Day, (Urbana, Ill.)

6046:

Slugin, S. N. On the theory of Newton's method and Chaplygin's method. Dokl. Akad. Nauk SSSR 120 (1958), 472-474. (Russian)

In Newton's method the approximations of a solution of $P(x) = 0$ are obtained by the algorithm $x_{n+1} = x_n - \Gamma_n^{-1} P(x_n)$, where in some sense Γ_n is close to the derivative $P'(x_n)$. The author considers the problem in a Banach space, replaces Γ_n^{-1} by L , and places conditions on L that assure the convergence of the successive approximations $x_{n+1} = x_n - LP(x_n)$ to a solution of $P(x) = 0$. It is not necessary that L have an inverse, and the conditions on L are said to lead to selections of operators L that are convenient for computation. When the space is a complete vector lattice he gives a uniqueness theorem for the problem $P(x) = 0$ and gives a pair of algorithms which yield successive approximations which converge monotonically from below and above to a solution of $P(x) = 0$. This is called a Chaplygin type method. This adds to what is now a large variety of such results, and it is the reviewer's opinion that what is needed most at the moment is a critical study of the existing algorithms.

J. P. LaSalle (Baltimore, Md.)

6047:

Citanadze, E. S. Investigation of a functional analogue of a Lichtenstein non-linear integral equation. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 650-653. (Russian)

As an analogue of the Lichtenstein integral operator, the author considers the functional

$$F(x) = (n+1)^{-1} \sum a_{i_1 \dots i_n} \varphi(x_{i_1}) \dots \varphi(x_{i_n}),$$

defined for $x = (x_i)$ in the unit sphere S_1 of l_2 , where the φ

are functions twice differentiable on $(-1, 1)$, n is a natural number, the a 's are real coefficients, $\sum a_{i_1 \dots i_n}^2 < \infty$ and $\sum [\varphi(x_i)]^2 \leq M^2$ if $x \in S_1$. In these circumstances the series for $F(x)$ is uniformly convergent on S_1 and defines a weakly continuous functional which is strongly differentiable: its strong derivative generates the functional $L_F x$; $L_F x$ is the element of l_2 with i_0 th component

$$\varphi'(x_0) \sum a_{i_1 \dots i_n} \varphi(x_{i_1}) \dots \varphi(x_{i_n}).$$

$L_F x$ is weakly continuous in S . If the $a_{i_1 \dots i_n}$ are non-negative and symmetric in the i 's and the φ are even and nonnegative, $\varphi(0)=0$, then L_F has an infinite number of distinct proper vectors. J. L. B. Cooper (Cardiff)

6048:

Vainberg, M. M.; and Šragin, I. V. The Nemyckii operator and its potential in Orlicz spaces. Dokl. Akad. Nauk SSSR 120 (1958), 941-944. (Russian)

Given B a measurable subset of a finite-dimensional Euclidean space, a Young function M defines the Orlicz space $L^M = L^M(B)$, the space L_M of functions such that $M(|u(x)|)$ is summable over B , and the space L_{M^*} of functions u such that $M(k|u(x)|)$ is summable for arbitrary k . For $g(u, x)$ on $(-\infty, \infty) \times B$, the operator h is defined by $(hu)(x) = g(u(x), x)$. Conditions are given in order that this operator map any one of the spaces L^M, L_M, L_{M^*} onto one of the corresponding spaces defined for a second Young function M_1 , and the continuity of the operator is discussed. If the operator h is on L^M to L^{M_1} , then it is the gradient of the functional f , where $f(u) = \int_B dy \int_0^{u(y)} g(v, y) dy$. J. L. B. Cooper (Cardiff)

CALCULUS OF VARIATIONS

See 5929, 6204.

GEOMETRIES, EUCLIDEAN AND OTHER

See also 5790.

6049:

Goormaghtigh, R. Théorèmes récurrents relatifs aux polygones inscriptibles. Mathesis 66 (1957), 288-291.

Let A_1, A_2, \dots, A_n be points on the unit circle of the complex plane, t_1, t_2, \dots, t_n the corresponding complex numbers. Let $P_{v,1}, P_{v,2}, P_{v,3}, \dots, P_{v,v}$ be the points corresponding to the elementary symmetric functions

$$\sum t_i, \sum t_i t_j, \sum t_i t_j t_k, \dots, \sum t_1 t_2 \dots t_v$$

of $t_1, t_2, \dots, t_v, 1 \leq v \leq n$. The author derives relations between the points $P_{v,j}$ ($1 \leq v \leq n, 1 \leq j \leq v$) and interprets geometrically the meaning of the points $P_{v,1}, P_{v,2}$ and $P_{v,3}$ with respect to the polygon $A_1 A_2 \dots A_v$ (and to the point 1).

The point $P_{v,1}$ ($v \geq 3$) is called orthocenter of the polygon $A_1 A_2 \dots A_v$ and is determined by the following properties: (1) $P_{3,1}$ is the orthocenter of the triangle $A_1 A_2 A_3$; (2) $P_{v,1}$ is the common intersection point of v unit circles with centers in the orthocenters of the polygons

$$A_2 A_3 \dots A_v, A_1 A_3 \dots A_v, \dots, A_1 A_2 \dots A_{v-1}.$$

Similar but more complicated recurrent interpretations are given for the points $P_{v,2}$ and $P_{v,3}$.

M. Fiedler (Prague)

6050:

Venkatachaliengar, K. An elementary proof of Morley's theorem. Amer. Math. Monthly 65 (1958), 612-613.

6051:

Mandan, Sahib Ram. Spheres associated with a semi-orthocentric tetrahedron. Res. Bull. Panjab Univ., no. 127 (1957), 447-451.

In each face of a semi-orthocentric tetrahedron the author considers two circles $(g_i), (l_i)$ ($i=a, b, c, d$) coaxial with the circumcircle (o_i) and the ninepoint circle (n_i) (i.e., the Griffiths pencil) of that face. The center of (g_i) coincides with the centroid G_i of the face considered, while G_i and the orthocenter H_i of that face are diametrically opposite points on the circle (l_i) .

The four circles (g_i) lie in pairs on two spheres, and the same holds for the four circles (l_i) . The author thus associates with the semi-orthocentric tetrahedron two new pairs of spheres, which form two coaxial pencils with the known spheres associated with that special tetrahedron.

The author observes that the latter two pencils of spheres coincide if the tetrahedron considered is orthogonal (i.e., orthocentric), and he points out a number of properties of the spheres of that "united" pencil. Nearly all of those properties, however, have been anticipated [cf. Nathan Altshiller-Court, Modern pure solid geometry, New York, 1935; p. 264, art. 805ff.].

N. A. Court (Norman, Okla.)

6052:

Kolobov, P. G. Investigation of configurations of Chasles. Rostov. Gos. Ped. Inst. Uč. Zap. 4 (1957), 53-59. (Russian)

Soient $A_{ij\alpha}$ ($i, j=1, \dots, 4; \alpha=1, 2$) les points d'intersection de la quadrique Q avec l'arête $A_i A_j$ d'un quadrilatère et considérons les quatre plans $\alpha_i = A_{i1\alpha} A_{i2\alpha} A_{i3\alpha}$ (α_i et α_j n'ayant pas un point $A_{r\alpha\beta}$ en commun); alors les droites $\alpha_i \cap A_j A_k A_l$ sont situées sur une quadrique (théorème de Chasles). L'A. étudie la position mutuelle des quadriques (en nombre 64) ainsi engendrées et il prouve: si β_i est le plan polaire du point A_i par rapport au Q , alors les quatre droites $\beta_i \cap A_j A_k A_l$ sont situées sur une quadrique. A. Svec (Prague)

6053:

Benz, Walter. Axiomatischer Aufbau der Kreisgeometrie auf Grund von Doppelverhältnissen. J. Reine Angew. Math. 199 (1958), 56-90.

Für Verf.s Aufbau der Kreisgeometrie ist kennzeichnend, dass er den Berührsatz fallen lässt (d.h. die Forderung, zu einem Kreise K , einem Punkt p auf K und einem Punkt ausserhalb K gebe es genau einen Kreis durch q , der K in p berührt), ferner auch früher und weitgehende Arbeiten mit abstrakten Doppelverhältnissen. Gegeben sind zwei abstrakte Mengen \mathfrak{P} (die der Punkte) und Θ (die der Werte), ferner eine Abbildung σ der Menge der geordneten Punktquadrupel (aus verschiedenen Punkten) in Θ . In Θ ist eine Teilmenge Δ ausgezeichnet. Unter dem Kreis (abc) versteht man die Menge, die ausser a, b, c alle x mit $\sigma(abcx) \in \Delta$ enthält. [Diesen Ansatz, aber spezieller, findet man auch bei H. Freudenthal, derselbe J. 194 (1955), 190-192; MR 16, 1145; und bei L. Peczar, Monatsh. Math. 54 (1950), 210-220; MR 12, 524.] Es werden nun Forderungen gestellt: $KQ1$, aus $\sigma(abcd) \in \Delta$, $\sigma(abde) \in \Delta$ folgt $\sigma(abce) \in \Delta$ (wenn a, b, c, d, e verschieden sind); $KQ2$, aus $\sigma(abcd) \in \Delta$ folgen $\sigma(abdc)$, $\sigma(acbd)$, $\sigma(bacd) \in \Delta$. $KQ1-2$ kennzeichnen die

"allgemeinen Kreisquaternare"; geometrisch bedeuten sie ungefähr, dass durch drei Punkte genau ein Kreis geht. Forderungen, unabhängig von Δ , lauten: Q_0 , $\sigma(abcx) = t \in \Theta$ besitzt im wesentlichen eine Lösung x ; Q_1 , aus $\sigma(abcd) = \sigma(a_1b_1c_1d_1)$, $\sigma(abde) = \sigma(a_1b_1d_1e_1)$ folgt $\sigma(abce) = \sigma(a_1b_1c_1e_1)$ (für verschiedene a, b, c, d, e usw.); Q_2 , aus $\sigma(abcd) = \sigma(a_1b_1c_1d_1)$ folgt $(\alpha) \sigma(bacd) = \sigma(b_1a_1c_1d_1)$, $(\beta) \sigma(acbd) = \sigma(a_1c_1b_1d_1)$, $(\gamma) \sigma(abdc) = \sigma(a_1b_1d_1c_1)$. Verf. spricht, je nachdem welche Forderungen erfüllt sind, von α -Quaternaren usw. Hinzu kommt noch die Eigenschaft $(\pi) \sigma(abcd) = \sigma(dcba)$. Von Interesse sind insbesondere die $\gamma\pi$ -(Kreis)-Quaternare, in denen ein Doppelverhältniskalkül möglich ist und in denen u.a. die Identität von Peczar und der Satz von Miquel gilt. In $\beta\gamma\pi$ -Kreis-Quaternaren kommt man weiter zu einer Winkeldefinition und -rechnung. Die Stellung des Berührsatzes wird untersucht. Man verstehe unter \mathfrak{B} die Konjunktion der Forderungen \mathfrak{B}_1 "die Winkelgleichheitsrelation hat Äquivalenzeigenschaften", \mathfrak{B}_2 "Winkel sind beliebig abtragbar", \mathfrak{B}_3 "Sehnentangentenwinkelsatz", \mathfrak{B}_4 "Satz von der Winkelsumme im Dreieck". Dann ist z.B. der Berührsatz unabhängig von der Konjunktion der "trivialen" Kreisgeometrieaxiome, der Gruppe \mathfrak{B} , dem Büschelsatz und dem Miquelschen Satz. *H. Freudenthal* (Utrecht)

6054:

Černyaev, M. P. The method of M. Chasles, modified by Professor V. Ya. Činger, for constructing a plane curve of third order through nine given points. Rostov. Gos. Ped. Inst. Uč. Zap. 4 (1957), 35-41. (Russian)

The author presents some classical properties and methods of construction of the plane cubic. It is not pointed out which part of them is to be credited to V. Ya. Činger (1836-1907), nor where and when his contribution was published. Neither does the paper contain any other historical or bibliographical references, although the paper is of a historico-methodological nature, according to the sub-title. *N. A. Court* (Norman, Okla.)

6055:

Kijne, D. The axiom of Archimedes and the theorem of Mohr-Mascheroni. Simon Stevin 31 (1957), 97-111. (Dutch)

In his thesis on plane construction field theory [Van Gorcum, Assen, Netherlands, 1956; MR 18, 501] the author has carefully discussed problems in elementary constructions arising from phrases like "take an 'arbitrary' point", etc. This theory is applied here to the rôle played by the Archimedes axiom in the theorem: all elementary constructions can be carried out by making use of compasses only. *O. Bottema* (Delft)

6056:

Erdős, Pál. On some geometrical problems. Mat. Lapok 8 (1957), 86-92. (Hungarian)

The paper gives lots of geometric problems and conjectures of combinatoric type concerning mainly the number of different distances between n points and the number of certain curves and surfaces (straight lines, circles, planes, hyperplanes) going through two, three, etc., points of given sets. It contains many references to results obtained previously (many of them by the author); also a few proofs or sketches of proofs are given. [It is somewhat disturbing that at some places references to footnotes and those to formulas are confounded. The paper is written in an agreeable conversational style.] *J. Aczél* (Debrecen)

6057:

Zimmer, Hans-Georg. Über Quadrate der affinen Rechtwinkelgeometrie. Math. Ann. 135 (1958), 340-351.

With the assumption of Fano's axiom and the 'little' Desargues theorem, the author investigates the consequences of introducing an absolute involution in an affine plane. After studying the 'geometry of squares', coordinatization based on a Veblen-Wedderburn system yields the appropriate analytical geometry. *G. de B. Robinson* (Toronto, Ont.)

6058:

Burde, Gerhard. Der Satz von Desargues in der Moufang-Ebene. Math. Ann. 135 (1958), 352-353.

If parallelism and perpendicularity are defined analytically in a Moufang plane as in ordinary Euclidean geometry then the general theorem of Desargues follows from its specialization. The reference to Forder, "Coordinates in geometry" [Auckland Univ. Coll. Bull. (1953); MR 15, 644] is significant for this and the preceding paper [6057 above]. *G. de B. Robinson* (Toronto, Ont.)

6059:

Bernhart, Arthur. Curves of pursuit. II. Scripta Math. 23 (1957), 49-65 (1958).

Conclusion of the survey begun in part I [same Scripta 20 (1954), 125-141; MR 16, 513].

6060:

Krokos, S. I. Movable models of a one-sheeted hyperboloid. Kiv. Derž. Univ. Nauk Zap. 16 (1957), no. 2 = Kiev. Gos. Univ. Mat. Sb. 9 (1957), 115-118. (Russian)

Two easily constructed models of string in a movable wooden framework, illustrating certain affine transformations.

6061:

Ionescu-Bujor, Constantin. Sur certains groupes de transformations. Bul. Inst. Politehn. București 19 (1957), no. 3/4, 15-26. (Romanian. Russian and French summaries)

The group is formed by the homologies $\omega_p(O, \lambda)$ in projective space E_p expressed by the equations

$$\rho \frac{y_i}{\alpha_i} = \frac{x_1}{\alpha_1} + \dots + \frac{x_{i-1}}{\alpha_{i-1}} + \lambda \frac{x_i}{\alpha_i} + \frac{x_{i+1}}{\alpha_{i+1}} + \dots + \frac{x_{p+1}}{\alpha_{p+1}} \\ (i = 1, 2, \dots, p+1),$$

with double point $O(\alpha_1, \alpha_2, \dots, \alpha_{p+1})$ and with

$$\frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} + \dots + \frac{x_{p+1}}{\alpha_{p+1}} = 0$$

as plane of double points. The set $[\omega_p]$ of these ω_p contains the identity $\omega_p(O, \infty)$, the inverse $\omega_p^{-1}(O, \lambda) = \omega_p(O, 1-p-\lambda)$ and a multiplication

$$\omega_p(O, \lambda_2) \omega_p(O, \lambda_1) = \omega_p(O, (\lambda_1 \lambda_2 + p)/(\lambda_1 + \lambda_2 + p - 1)),$$

and thus forms a commutative group, which also, for $\lambda = \frac{1}{2}(1-p)$, contains an involution of order 1. This group is further investigated. *D. J. Struik* (Cambridge, Mass.)

6062:

Ionescu-Bujor, Constantin. Au sujet des invariants d'un groupe de transformations. Bul. Inst. Politehn. București 19 (1957), no. 3/4, 27-30. (Romanian. Russian and French summaries)

Under the group $\omega_p(O, \lambda)$, defined in the previous article [6061], the ratios of the determinants of the

matrix

$$\begin{pmatrix} x_1 x_2 \cdots x_{p+1} \\ \alpha_1 \alpha_2 \cdots \alpha_{p+1} \end{pmatrix}$$

are invariants. This result is interpreted geometrically in terms of cross ratios. *D. J. Struik* (Cambridge, Mass.)

6063:

Gutmann, Marcian. Quelques remarques sur un groupe de transformations à un paramètre sur l'espace Z_n . *Bul. Inst. Politehn. București* 19 (1957), no. 3/4, 31-35. (Romanian. Russian and French summaries)

The group $\omega_p(O, \lambda)$, defined above [#6061], can be transformed into a group expressed by

$$y = tx + (e \cdot x)e,$$

where e is a constant vector. The space may be complex. It is shown how certain relations based on this equation can be generalized to more general vector spaces.

D. J. Struik (Cambridge, Mass.)

6064:

Hu, Hoo-sung. A characterization of a projective flat space. *Acta Math. Sinica* 8 (1958), 269-271. (Chinese. English summary)

6065:

Ostrom, T. G. Correction to "Transitivity in projective planes". *Canad. J. Math.* 10 (1958), 507-512.

The author's proof of theorem 16 (ii) [same *J.* 9 (1957), 389-399; MR 19, 445] is in error. (The error was detected by G. Pickert.) The near-field plane of order 9 and its dual are counterexamples. This paper gives the corrected version of theorem 16 (ii), excluding $n=9$.

H. J. Ryser (Columbus, Ohio)

6066:

Valette, Guy. Le plan conforme sur le corps à trois éléments. *Mathesis* 66 (1957), 269-283.

The author states that, although finite projective geometries were developed by Veblen and Young in 1906, he knows of no similar treatment in conformal geometry. In this paper he studies the conformal transformation of a body of three elements on a plane.

T. R. Holcroft (Aurora, N.Y.)

6067:

Fedorova, R. N. Isotopy of surfaces of second order in the geometry of Lobachevskii. *Soobšč. Akad. Nauk Gruz. SSR* 20 (1958), 137-142. (Russian)

The author states that the classification of Coolidge and Bromwich of quadratic surfaces in hyperbolic space is not complete, first gives a list of 43 kinds, 39 of which are specified by canonical equations, and then shows that these kinds partly coincide for infinitesimal transformations, giving rise to 17 different kinds.

As the author states what she means by metrical equivalence, the reviewer should like to accentuate that the method of classification used is different from that of Coolidge, who obtained 34 different kinds. Coolidge classifies by considering the reality of the curve of intersection of the quadric with the absolute, the reality of the developable of common tangent planes and requires that the quadric should have real points "inside the absolute".

J. A. Barrau, in his excellent book "Analytische Meetkunde" [1918/1927, unfortunately only accessible in Dutch], gave a complete real classification of pairs of quadrics in projective space, differentiating also according to the signature of the quadrics. From this classification it follows that, according to the reality of the roots of the

characteristic equation of the pencil and the signature of the surfaces only, there are 76 real projective different kinds of pairs of quadrics. If one of the quadrics is to have a signature 4 (elliptic geometry) there are 7 kinds left. If one of them has the signature 2 (hyperbolic geometry of Lobatschewsky), there are 36 pairs left. But for these, points inside the absolute are not necessarily real, e.g., in the obvious case of signature 0, type [(111)1], the double-tangent one-sheet hyperboloid is completely outside the absolute sphere. Finally there are 34 kinds left. So, more than 34 kinds could be obtained only from other methods for classification. In a complete metrical classification the kinds for which, e.g., in the type [2, 1, 1], the common tangent plane of quadric and absolute is intersected in harmonic pairs ["tangents of the node in the curve of intersection perpendicular"] should be distinguished, not only those for which the pencil involution in the tangent plane is identical [the surfaces of revolution]. As the author did not include in her list such kinds as $((x-k)/a)^2 + y^2/b^2 - z^2/b^2 = 1$, $0 < k < 1$, $a+k=1$, she must have used a classification disregarding these differences, which accounts for the absence of several kinds of quadrics. According to the arrangement of the canonical forms in her list, she intended to subdivide the pairs according to sign and magnitude of the roots of the characteristic equation of the pencil, one of which can always be chosen arbitrarily, e.g. unity. Then, however, two kinds are still lacking in her list:

$$[1, 1, 1, 1], 0 < \lambda a < b, 1 < a, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-\lambda)^2}{1+\lambda^2} = 1,$$

$$[(1, 1), 1, 1], a < \lambda, -\frac{x^2}{a^2} - \frac{y^2}{a^2} + \frac{(z-\lambda)^2}{1+\lambda^2} = 1.$$

E. M. Bruins (Amsterdam)

CONVEX SETS AND DISTANCE GEOMETRIES

See also 5768, 5882, 6026, 6112, 6179.

6068:

Fan, Ky. On the equilibrium value of a system of convex and concave functions. *Math. Z.* 70 (1958), 271-280.

Generalizations of some known results on nonnegative matrices: S is the $(n-1)$ -dimensional simplex in Euclidean space E^n of points $x = (x_1, \dots, x_n)$ satisfying $x_i \geq 0$, $\sum x_i = 1$; S_1 the face on which $x_1 = 0$; and the functions $\{f_1, \dots, f_n; g_1, \dots, g_n\}$ satisfy: (a) Each f_i is continuous and convex on S ; (b) each g_i is continuous, concave, and positive on S ; (c) $f_i(x) \leq 0$ for $x \in S_1$; (d) for each $x \in S$ there is an index i for which $f_i(x) > 0$. The problem concerns the existence of an equilibrium point $x \in S$ satisfying

$$g_i(x) = \lambda f_i(x) \quad (1 \leq i \leq n).$$

The first theorem establishes existence and unicity, with a minimax characterization of λ^{-1} ; the second gives a further characterization

$$\lambda^{-1} = \max_{\alpha_i > 0} \min_{\sum \alpha_i = 1} \sum \alpha_i f_i(x) / \sum \alpha_i g_i(x).$$

Theorem 3 states that for any $\gamma_i \geq 0$, $\sum \gamma_i = 1$, the equations $\rho f_i(x) - g_i(x) = \beta \gamma_i$ can be satisfied with some point x and scalar $\beta > 0$ if and only if $\rho > \lambda$. Finally, theorem 4 relates λ to the equilibrium value λ_n of a subsystem (analogous to that of a principal submatrix), showing

$\lambda > \lambda_n$, and stating that $\lambda > \rho > \lambda_n$ if and only if there exists a point x in the interior of S such that

$$\rho f_i(x) - g_i(x) = 0 \quad (1 \leq i \leq n-1),$$

$$\rho f_n(x) - g_n(x) < 0.$$

A number of corollaries develop in more detail parallels with the linear theory.

A. S. Householder (Oak Ridge, Tenn.)

6069:

Pták, Vlastimil. On the absolutely convex envelope of a set in a finite dimensional vector space. *Časopis Pěst. Mat.* 83 (1958), 343-347. (Czech. Russian and English summaries)

Let A be a compact subset of an n -dimensional vector space. Then the symmetrical convex envelope of A is compact and consists of all vectors of the form $\sum_{i=1}^n \lambda_i a_i$, where $a_i \in A$ and the numbers λ_i fulfill the inequality $\sum_{i=1}^n |\lambda_i| \leq 1$.

From the author's summary

6070:

Erdős, Pál; und Vincze, István. Über die Annäherung geschlossener, konvexer Kurven. *Mat. Lapok* 9 (1958), 19-36. (Hungarian. Russian and German summaries)

In this clearly written and comprehensible paper, the authors give, after an expository introduction on minimal circumscribed and maximal inscribed circles of convex curves and about their distances and parallel-curves, a new proof of the unicity of the minimal annulus containing a convex curve (and having its center inside the curve). This proof is based on a remark of H. Lebesgue. Finally, the authors show that an equilateral triangle cannot be approximated arbitrarily by Tschirnhaus-curves with n foci (thus solving a problem of E. Vázsonyi), but they give an example of a convex curve containing one straight segment for which such an approximation is possible and they ask whether such convex curves containing two straight segments exist. (It is a pity that there are no illustrative figures in the paper. Sometimes also the lack of brackets enclosing the references is somewhat disturbing.)

J. Aczél (Debrecen)

6071:

Valentine, F. A. A three point convexity property. *Pacific J. Math.* 7 (1957), 1227-1235.

The property in question (called P_3) is the following: A set S in a linear space has property P_3 if, for every three points x, y, z in S , at least one of the straight line segments xy, xz, yz lies in S . Property P_3 is obviously weaker than the statement that S is the union of two convex sets. In this note it is shown that, in the plane (E_2), a set S which has property P_3 , but is not the union of two convex sets, is the union of three convex sets with a nonempty intersection, and contains an odd number, greater than 1, of points of local nonconvexity. These properties are, however, not sufficient for P_3 , and no complete characterization is attained.

For general E_n , it is shown that if S has property P_3 , then it is starshaped with respect to all points of local nonconvexity. This observation proves useful in a more recent paper by the same author [6072 below].

E. G. Straus (Los Angeles, Calif.)

6072:

Valentine, F. A. The intersection of two convex surfaces and property P_3 . *Proc. Amer. Math. Soc.* 9 (1958), 47-54.

The writer has previously defined the 3-point property P_3 [6071 above], generalizing convexity for a set S , as the assertion that for any triple $\{x_i | i=1, 2, 3\}$ in S some

segment $[x_i, x_j]$ CS. The author uses the definition: a point p in S is a locally convex point if for some neighborhood N of p and every pair $x, y \in S \cap N$, the segment $[x, y]$ CS. Otherwise p is a point of local non-convexity and the collection of points of local non-convexity is denoted by Q . In topological linear spaces, if S satisfies P_3 , if its linear extension is at least 3-dimensional, and if Q has an isolated point, then Q has at most 2 points. An interesting result is that if S is a compact subset of a finite dimensional normed linear space, and if the convex kernel of S has an inner point, and if Q is contained in the open part of the convex hull of S , then Q consists of a finite number of disjoint $(n-2)$ -dimensional manifolds.

D. G. Bourgin (Urbana, Ill.)

6073:

Ohshio, Shigeru. On the explicit representations of the isoperimetric deficiencies and the Brunn-Minkowski's theorem for inner parallel surfaces. *Tensor (N.S.)* 8 (1958), 38-54; erratum 9 (1959), 142.

The paper gives explicit integral representations of isoperimetric deficiencies and sketches, in an elementary way, proofs which, for linguistic reasons, are not always easy to follow. For instance, on p. 45, critical values are enumerated in increasing order, on the grounds that they constitute a countable set.

L. C. Young (Cambridge, England)

6074:

Froda, Alexandre. Espaces p -métriques et leur topologie. *C. R. Acad. Sci. Paris* 247 (1958), 849-852.

The author defines, for each set of $p+1$ points of a given space, a real number which he calls the p -distance associated with this set of points. Axioms are set up on this p -distance which are quite similar to those associated with the usual metric (to which this pseudo metric reduces in the case $p=1$). Various topological properties of these (pseudo) p -metric spaces are studied.

D. W. Hall (Endicott, N.Y.)

6075:

Froda, Alexandre. Points associés d'une fonction abstraite. *C. R. Acad. Sci. Paris* 247 (1958), 901-903.

The author extends a property of abstract functions defined on a structureless support and taking their values in a metric space to the case in which the value space is (pseudo) p -metric [see the review above]. Except for an exceptional set of support, one can associate with any p distinct points another point such that the values of the function at these $p+1$ points are, in their aggregate, as close together as one desires in the sense of the p -metric.

D. W. Hall (Endicott, N.Y.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 5845.

6076:

Lelek, A. Sur l'unicohérence, les homéomorphismes locaux et les continus irréductibles. *Fund. Math.* 45 (1957), 51-63.

Let X and Y be two compact metric spaces and f a function such that $f(X)=Y$. The function f is said to be a local homeomorphism in the large sense when there exists, for each point $x \in X$, a neighborhood U_x such that the partial function $f|U_x$ is a homeomorphism. When these U_x , for each $x \in X$, can additionally be so taken that their images $f(U_x)$ are open subsets of Y containing

$f(x)$, the function f is simply called a local homeomorphism. Finally, f is called a recouvrement of Y by X when there exists, for every point $y \in Y$, a neighborhood V_y such that the point set $f^{-1}(V_y)$ is the sum of a family of disjoint open sets, $f^{-1}(V_y) = \sum_i U_i$, such that upon the latter the partial functions $f|U_i$ are homeomorphisms and $f(U_i) = V_y$. These three classes of functions are by definition continuous; every local homeomorphism is trivially one in the large sense, but not conversely (example); every recouvrement is also trivially a local homeomorphism, but not conversely (example).

The author shows that a function f is a local homeomorphism if and only if it is one in the large sense and simultaneously is an interior transformation. Using the usual notion of order at a point, it is shown that if X is compact and f is a local homeomorphism in the large sense, then $\text{ord}_x X \leq \text{ord}_{f(x)} f(X)$. This result is employed in proving that every local homeomorphism in the large sense which transforms a continuum X into a dendrite is a homeomorphism.

Turning next to irreducible continua and the notion of unicoherence it is shown that if all of the tranches of adhesion of a continuum X irreducible between two points a and b are unicoherent, then X is likewise unicoherent. By example the converse is not true. Concerning finally the composants, $C_a(X)$, of points a of a continuum X , it is shown that if X is irreducible between a and b , $C = C_a(X)$ is open (in X), and φ is a homeomorphism transforming C into a subset of a compact space Y , then the closure $\overline{\varphi(C)}$ is an irreducible continuum between $\varphi(a)$ and every point $q \in \overline{\varphi(C)} - \varphi(C) \neq \emptyset$. The following problem is raised: Is it true that every composant (of dimension $n \geq 1$) of any indecomposable continuum is homeomorphic with a dense subset of a euclidean region (of dimension $n+1$)?

W. W. S. Claytor (Washington, D.C.)

6077:

Whyburn, G. T. Topological characterization of the Sierpiński curve. *Fund. Math.* 45 (1958), 320-324.

The following statements about a plane locally connected 1-dimensional continuum S are equivalent: (1) The boundaries of the complementary domains of S are simple closed curves, no two of which meet; (2) S has no local separating points; (3) S is homeomorphic to Sierpiński's "universal plane curve" [C. R. Acad. Sci. Paris 162 (1916), 629-632]. The author had obtained these results in 1930 [*Ann. Soc. Polon. Math.* 9, 172] using long and complicated arguments (unpublished); the present proofs are short and relatively simple. The main step is to prove that two spaces S, S' satisfying (1) are homeomorphic, by constructing a sequence of isomorphic subdivisions of them; the details are simplified by the device of modifying S, S' (via upper semicontinuous decompositions) so that most of their complementary domains are replaced by single points. {There are misprints in the last sentence of the paper; for [4] read [6], and delete the subsequentpagenumbers.}

A. H. Stone (Manchester)

6078:

Kowalsky, H.-J. Kennzeichnung von Bogen. *Fund. Math.* 46 (1958), 103-107.

Let E be a non-degenerate, connected, locally connected, separable T_1 -space. The author proves two theorems: (1) in order that E be homeomorphic with a closed, half-open, or open number interval it is necessary and sufficient that some two of every three connected proper subsets of E fail to cover E ; and (2) in order that E be an

arc it is necessary and sufficient that E be irreducibly connected between some two of its points; (2) being an easy corollary of (1). The methods of proof are elementary and quite similar to those used in Chapters I and II of R. L. Moore's "Foundations of point-set theory" [Amer. Math. Soc. Colloq. Publ., New York, N.Y., 1932].

Characterizations of the above type are known for Hausdorff spaces. In fact, if E satisfies the sufficiency condition in (1), every connected open set has at most two boundary points. (If a connected open set U had three boundary points, p_1, p_2 , and p_3 , then the components of $E - p_1$ containing U would violate the condition.) So for Hausdorff spaces (1) follows from results in the reviewer's paper [Bull. Amer. Math. Soc. 45 (1939), 623-628; MR 1, 45].

F. B. Jones (Chapel Hill, N.C.)

6079:

Grace, Edward E. Totally nonconnected im kleinen continua. *Proc. Amer. Math. Soc.* 9 (1958), 818-821.

Upon removing the regularity built into the ordinary definition of connectedness im kleinen, one gets (for a connected topological space T) the following: If T is not connected im kleinen at a point p , there exists an open set U containing p such that p is a boundary point of the p -component of U . Now if U is an open set and T is connected im kleinen at no point of U , it is not necessarily true that each point of U is a boundary point of the component of U which contains it (e.g., when $U=T$). However, by suitably restricting T in a category sense, U contains such an open set V which is dense in U (i.e., every component C of V is a subset of the closure of $T-C$).

The author goes further and obtains in considerable generality a theorem which, for a connected, separable, complete metric space, reduces to the following: If T is connected im kleinen at no point of an inner-limiting ($=G_\delta$) subset of U which is dense in U , then there exists an open subset V of U which is dense in U such that every component of V is a subset of the closure of its complement. This conclusion will not follow from the supposition that T is locally connected at no point of U . The counter example is a bounded plane continuum T such that (taking $U=T$) if V is a (relatively) open subset of T with $\overline{V}=T$, then some component of V contains an interior point (relative to T) even though T is locally connected at none of its points.

The author's methods and concepts are essentially those of a previous paper [Proc. Amer. Math. Soc. 6 (1958), 98-104; MR 20#1960]. He generalizes the notion of inner-limiting (G_δ) set in a natural way relative to the smallest number of open sets which form a topological basis. His definition of the term "Baire topological" on an open set as a generalization of the 2nd category notion results in the following: If the connected topological space T is Baire topological on the open set U , then U contains an open set V such that T is Baire topological on every open subset of V . This theorem and the notions involved are reminiscent of and possibly related to those in Dunford's paper [Ann. of Math. (2) 41 (1940), 639-661; MR 2, 72]. The reviewer is of the opinion that this area of general topology is deserving of further study.

F. B. Jones (Chapel Hill, N.C.)

6080:

Alzenberg, L. A neighborhood theorem of plane topology. *Moskov. Oblast. Pedagog. Inst. Uč. Zap.* 57 (1957), 199-206. (Russian)

For an $\varepsilon > 0$ let M_1, M_2, \dots be a sequence of continua

covering the plane E_2 , each of diameter $< \varepsilon$, each bounded by a simple closed curve and such that for every i and ε the set $\overline{M_i \cap \bigcup_j M_{k_j}}$ ($j=1, 2, \dots, e$) is connected; one supposes that some plane square of length $\geq 19\varepsilon$ intersects finitely many of the sets M_i . One says then that the covering is "partially regular". One gives an elementary proof of the following theorem: For every $\varepsilon > 0$ and every partially regular covering C of the plane E_2 there exists an element $X \in C$ intersecting at least 6 other members of C [cf. L. M. Lihtenbaum, *Mat. Sb.* 1(43) (1936), 907-916; A. D. Aleksandrov, *Mat. Sb.* 2(44) (1937), 307-318; O. V. Lokucievskii, *Uspehi Mat. Nauk* 5 (1950), no. 5(39), 165-167; *MR* 12, 519].

D. Kurepa (Zagreb)

6081:

*Whyburn, Gordon Thomas. *Topological analysis*. Princeton Mathematical Series. No. 23. Princeton University Press, Princeton, N. J., 1958. xii+119 pp. \$4.00.

L'objet de ce livre est selon l'expression de l'auteur "les propriétés fondamentales de l'analyse et plus particulièrement de la théorie des fonctions d'une variable complexe, à caractères essentiellement topologiques exposés et démontrés entièrement par des méthodes topologiques ... en un mot, des théorèmes d'analyse établis par des démonstrations topologiques". Plusieurs livres ont été en tout ou partie consacrés à ce sujet: "Analytic topology" de Whyburn [*Amer. Math. Soc. Colloq. Publ.*, New York, N.Y., 1942; *MR* 4, 86], "Les principes topologiques..." de Stoilow [Gauthier-Villars, Paris, 1956; *MR* 18, 568], "Topological methods..." de Morse [Princeton Univ. Press, Princeton, N.J., 1947; *MR* 9, 20].

Les trois premiers chapitres sont consacrés aux éléments de topologie devant jouer un rôle important dans l'étude des applications analytiques: compacité, continus, continus localement connexes, tout cela dans des espaces métriques séparables; au chapitre III on trouve une démonstration du théorème de Jordan, ainsi que d'autres théorèmes sur la séparation des ensembles plans. Les notations de l'auteur sont restées celles de l'ouvrage précédent "Analytic topology" et le nouveau lecteur aurait sans doute préféré \cup au lieu de Σ et \cap au lieu de \cdot .

Le chapitre IV est consacré à un bref rappel élémentaire sur les fonctions d'une variable complexe et le chapitre V introduit l'outil essentiel de la suite du livre: l'indice de circulation. Cet indice est introduit par une représentation exponentielle de l'application donnée; on peut se poser la question de savoir ou limiter l'imbrication de l'analyse et de la topologie; l'auteur accepte cette représentation exponentielle mais n'a pas voulu utiliser la notion d'intégrale curviligne qui a permis par ailleurs une exposition très simple et très élégante de la notion d'indice de circulation, par L. V. Ahlfors dans son livre "Complex Analysis" [McGraw-Hill, New York-Toronto-London 1953; *MR* 14, 857]; c'est sans doute à ce choix que l'auteur fait allusion dans sa préface: "Ce choix des sujets a été grandement orienté par les recherches et les goûts particuliers de l'auteur". Ce choix étant fait l'intérêt de l'ouvrage réside dans l'usage intensif et prodigieusement fructueux de cette notion d'indice de circulation, qui avait déjà fourni à M. Morse le moyen d'établir des résultats fondamentaux sur le comportement global des applications intérieures.

Au chapitre VI l'indice de circulation est utilisé pour établir les deux caractères essentiels des fonctions analytiques reconnus par Stoilow, "lightness" (invariance des continus compacts non dégénérés) et "openness" (in-

variance des ouverts), d'où le théorème du maximum du module pour les applications intérieures. Les notions "light" et "open" peuvent s'étendre à des applications d'espaces métriques séparables localement compacts, et en particulier aux variétés à deux dimensions: l'image inverse d'un point est alors formée de points isolés; f est alors localement, topologiquement équivalente à $w=zs^k$ (k entier). L'étude de la dérivée pose encore des questions si l'on convient toujours de s'abstenir d'utiliser l'intégrale curviligne. Le chapitre VIII traite des degrés et des zéros d'une application différentiable et l'indice permet d'établir le théorème de Rouché ainsi que le théorème d'Hurwitz relatif à une suite de fonctions $f_n(z)$ convergeant uniformément vers $f(z)$ différentiable; l'hypothèse de différentiabilité de la fonction limite peut être abandonnée comme il résulte de l'étude faite au chapitre X. Le chapitre IX est consacré à une étude globale des applications "light", "open", "normal" des variétés compactes à deux dimensions et à l'établissement de relations du type d'Hurwitz concernant les caractéristiques d'Euler de ces variétés.

Tel qu'il est ce livre est du plus haut intérêt pour l'étude des caractères topologiques des applications intérieures des variétés à une dimension complexe et contribue à mettre en évidence le rôle joué par l'indice de circulation dans toutes ces questions; mais ce livre n'épuise pas les questions topologiques que posent les problèmes concernant $w=f(z)$ et pour lesquels d'autres méthodes topologiques ont dû être utilisées [cf. Collingwood, *Math. Z.* 67 (1957), 377-396; *Ann. Acad. Sci. Fenn. Ser. A. I.*, no. 250/6 (1958); *MR* 20#2449, #2451].

{On doit signaler à l'intention d'une seconde édition de cet ouvrage la nécessité de corriger de trop nombreuses confusions de lettres.}

L. Fourès (Marseille)

6082:

Gál, István S. On a generalized notion of compactness. I, II. *Nederl. Akad. Wetensch. Proc. Ser. A.* 60=Indag. Math. 19 (1957), 421-435.

The author calls a topological space (m, n) -compact if every open cover of cardinal $\leq n$ has a subcover of cardinal $\leq m$, (m, ∞) -compact if every open cover has a subcover of cardinal $\leq m$, and completely (m, x) -compact if every subspace is (m, x) -compact. Typical results: (1) Let X be a uniform space having a structure base of cardinal $\leq u$; if X is (m, u) -compact for some $m < u$, or if X is (m, n) -compact where $m < n$ and $u \leq n$, then X is (m, ∞) -compact. (2) Let X be a product space of $\leq m$ factors; if every finite subproduct is completely (m, ∞) -compact, then X itself is completely (m, ∞) -compact.

L. Gillman (Princeton, N.J.)

6083:

Gál, I. S. On the theory of (m, n) -compact topological spaces. *Pacific J. Math.* 8 (1958), 721-734.

Continuing his study of (m, n) -compact spaces [6082 above], the author defines (m, n) -filters and (m, n) -nets and characterizes (m, n) -compactness in terms of them; other notions considered are n -paracompactness and the uniform cardinality of a uniformizable space.

L. Gillman (Princeton, N.J.)

6084:

Kac, G. I. Functional closedness of completely regular spaces. *Dokl. Akad. Nauk SSSR* 120 (1958), 953-955. (Russian)

Theorem: A space G that admits a complete uniform structure is functionally-closed (i.e., a Q -space) if the discrete space whose cardinal is that of G is functionally-

closed. [This theorem was proved by T. Shirota [Osaka Math. J. 4 (1952), 23-40; MR 14, 395]. Indeed, Shirota's proof shows that it suffices to require only that the closed discrete subspaces of G be functionally-closed.]

M. Jerison (Princeton, N.J.)

6085:

Sklyarenko, E. On the imbedding of normal spaces into bicompaacts of the same weight and dimension. Dokl. Akad. Nauk SSSR 123 (1958), 36-39. (Russian)

The author proves by means of proximity structures that a normal space of covering dimension n can be embedded in an n -dimensional compact space, without increasing the weight (least cardinal of open sets); moreover, for any chosen countable family of closed sets A_k , it can be arranged that the dimension of the closure of A_k in the compactification is $\dim A_k$.

J. Isbell (Seattle, Wash.)

6086:

McAuley, Louis F. A note on complete collectionwise normality and paracompactness. Proc. Amer. Math. Soc. 9 (1958), 796-799.

Consider the following properties of a topological space: paracompactness, hereditary paracompactness, collectionwise normality, and hereditary collectionwise normality. In a Moore space (or a developable space) each one of these properties implies all of the others for the not-so-simple reason that every such space is metric. The author adds one more type of normality to the list: complete collectionwise normality (a combination of complete and collectionwise normality) and then proves that all five properties are equivalent in semimetric topological spaces even though such spaces need not be metric. Furthermore, in close approximations to a semimetric topological space (e.g., a Hausdorff space satisfying the first countability axiom) these conditions are not equivalent; in particular, collectionwise normality does not imply complete collectionwise normality.

It is shown that a semimetric topological space is F_σ -screenable (for every open covering G of the space there is a closed covering which refines G and which is the union of countably many discrete subcollections; in his argument the author should use the closures of the elements of his collections X_i). But a collectionwise normal F_σ -screenable topological space is paracompact. Hence, the above equivalences follow from results due to Bing, Dowker, and A. H. Stone.

With the help of one of Morita's theorems the author shows that a normal, pointwise paracompact, separable, semimetric topological space is completely collectionwise normal, paracompact, and hereditarily separable (such a space need not be perfectly separable, i.e., need not have a countable basis). As a corollary it follows that a normal, separable, Moore space is metrizable if it is pointwise paracompact. This is curiously parallel to the reviewer's theorem that a normal, separable, Moore space is metrizable if $2\kappa_1 < 2\kappa_2$. It is open to speculation whether there is a direct connection. It is also not known that separability is necessary and, in fact, no example of a semimetric topological space which is normal but not paracompact is known.

F. B. Jones (Chapel Hill, N.C.)

6087:

Smirnov, Yu. An example of zero-dimensional normal space having infinite dimensions from the standpoint of covering. Dokl. Akad. Nauk SSSR 123 (1958), 40-42. (Russian)

The author develops in some detail the properties of

C. H. Dowker's construction of example M [Quart. J. Math. Oxford Ser. (2) 6 (1955), 101-120; MR 19, 157]; in particular, it can be modified so as to yield a normal space X with $\text{ind } X = 0$ and $\dim X$ arbitrary.

J. Isbell (Seattle, Wash.)

6088:

Nagata, Jun-iti. On countable-dimensional spaces. Proc. Japan Acad. 34 (1958), 146-149.

Various necessary and sufficient conditions are given for a metrizable R to be countable-dimensional (i.e., the union of a countable number of 0-dimensional subsets). Some of them: (a) there exists a σ -locally finite open base \mathcal{V} such that the collection $\{V - V | V \in \mathcal{V}\}$ is point-finite; (b) there exists a closed continuous mapping f of a set $SCN(\Omega)$ onto R such that all $f^{-1}(p)$, $p \in R$, are finite; $N(\Omega)$ denotes the generalized Baire space (the product of countably many discrete spaces); (c) (for a space R with a σ -star-finite base): R is homeomorphic with a subset of $N(\Omega) \times R_\omega$, where R_ω is the subspace of the cube \mathcal{I}_ω consisting of points with almost all coordinates irrational.

M. Katětov (Prague)

ALGEBRAIC TOPOLOGY

See also 5940.

6089:

Lee, Ke-chun. Über die Eindeutigkeit von einigen kombinatorischen Invarianten endliches Komplexes. Sci. Record (N.S.) 1 (1957), 279-281.

Let K be any n -dimensional simplicial complex and let $a^i(K)$ denote the number of i -dimensional simplexes of K . In the first part of the note, the author gives a simple proof of a theorem due to W. Mayer [Ann. of Math. (2) 43 (1942), 594-605; MR 3, 318] which states that every combinatory invariant of the form

$$\delta(K) = \sum_{i=0}^n r_i a^i(K),$$

where the r_i 's are real numbers, must be equal to the Euler-Poincaré characteristic $\chi(K)$ multiplied by a constant k . In the second part of the note, a more complicated theorem about the combinatory invariants of K is proved.

Sze-tsen Hu (Detroit, Mich.)

6090:

Berikašvili, N. A. Axiomatic theory of group spectra. Soobšč. Akad. Nauk Gruz. SSR. 18 (1957), no. 6, 641-646. (Russian)

The concept of limit group of a direct [inverse] system of abelian groups is characterized here by 5 axioms. The axioms supplemented by some topological requirements characterize also the Chogoshvili limit of a direct system of compact groups. It is shown that if one of the axioms is dropped there do exist limit theories different from the classical ones.

W. T. van Est (Utrecht)

6091:

Berikašvili, N. A. On the axiomatic spectral theory and the duality rules for arbitrary sets. Akad. Nauk Gruz. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 24 (1957), 409-484. (Russian)

The first part of the paper gives a complete axiomatic treatment of the theory of group spectra and their limit groups, announced before [#6090 above]. The main result

of the second part is the duality of the homology groups $V_p(X, G)$ and $L_p(Y, H)$, where X, Y are arbitrary complementary subsets of S^n , G is a discrete group and H its character group, $V_p(X, G)$ the subgroup of the Vietoris homology group consisting of the classes with compact support, $L_p(Y, H)$ is similarly defined with respect to Y and H , the Vietoris homology of Y with coefficients in H being taken in the sense of Chogoshvili. Finally a duality theorem of Sitnikov is complemented by interpreting one of the groups occurring therein as a suitable homology group.

W. T. van Est (Utrecht)

6092:

Milnor, John. The Steenrod algebra and its dual. Ann. of Math. (2) 67 (1958), 150-171.

The paper deals with the Steenrod algebra \mathcal{S}^* , i.e., the algebra generated by the reduced powers of cohomology classes, with the iteration as the product. \mathcal{S}^* is defined, following Adem [in "Algebraic geometry and topology", Princeton Univ. Press, 1957; MR 19, 50; p. 191-238], as the quotient algebra of \mathcal{S}^* (free associative graded algebra generated by the reduced powers \mathcal{P}^i and by the δ associated with the coefficient sequence $0 \rightarrow Z_p \rightarrow Z_{p^2} \rightarrow Z_p \rightarrow 0$) by the ideal corresponding to relations valid for all cohomology classes of all complexes. (The case $p=2$ requires some slight modifications mentioned at the end of the paper.) A connected graded algebra A_* with unit (vector space over a field F) is a Hopf algebra if there exists a homomorphism of algebras $\phi_*: A_* \rightarrow A_* \otimes A_*$, with $\phi_*(a) = a \otimes 1 + 1 \otimes a + \sum b_i \otimes c_i$, $\dim b_i$ and $\dim c_i > 0$ when $\dim a > 0$. — Defining a dual algebra (by $A^* = A_{-n} = \text{Hom}(A_n \rightarrow F)$) and dual homomorphisms in the usual way, the dual A^* is again a Hopf algebra if A_* is of finite type. [Result stated without proof; the proof will be given in a paper by J. Milnor and J. Moore, to appear.]

The author proves that \mathcal{S}^* is a Hopf algebra by showing the existence of a (unique) ring homomorphism $\psi^*: \mathcal{S}^* \rightarrow \mathcal{S}^* \otimes \mathcal{S}^*$, for which he gives explicit formulas, and which is similar to the Cartan formula [C. R. Acad. Sci. Paris 230 (1950), 425-427; MR 12, 42] describing the action of Sq^i on a cup-product. — From the homomorphisms

$$\mathcal{S}^* \xrightarrow{\psi^*} \mathcal{S}^* \otimes \mathcal{S}^* \xrightarrow{\psi^*} \mathcal{S}^*$$

follow the dual homomorphisms

$$\mathcal{S}_* \xrightarrow{\psi_*} \mathcal{S}_* \otimes \mathcal{S}_* \xrightarrow{\psi_*} \mathcal{S}_*$$

in the dual Hopf algebra \mathcal{S}_* . H_* and H^* denoting homology and cohomology (with coefficients Z_p) of a finite complex, a homomorphism $\lambda_*: H_* \otimes \mathcal{S}_* \rightarrow H_*$ is defined, the value of $\mu\theta$ on α being the value of μ on $\theta\alpha$ for $\theta \in \mathcal{S}^*$, $\mu \in H_*$ and $\alpha \in H^*$. Its dual λ^* is a ring homomorphism of H^* into $H^* \otimes \mathcal{S}^*$.

The dual algebra \mathcal{S}_* is studied on the lens space $X = S^{2N+1}/Z_p$ (for N large), which can be considered as the $(2N+1)$ -skeleton of $K(Z_p, 1)$. Its cohomology group has generators α (for $H^1(X)$), $\beta = \delta\alpha$ (for $H^2(X)$) β^i for $H^{2i}(X)$ and $\alpha\beta^i$ for $H^{2i+1}(X)$ ($i \leq N$). Elements τ_k of \mathcal{S}_{2p-1} and ξ_k of \mathcal{S}_{2p-2} are defined by

$$\lambda^*\alpha = \alpha \otimes 1 + \beta \otimes \tau_0 + \beta^p \otimes \tau_1 + \cdots + \beta^p \otimes \tau_r,$$

$$\lambda^*\beta = \beta \otimes 1 + \beta^p \otimes \xi_1 + \cdots + \beta^p \otimes \xi_r.$$

The algebra \mathcal{S}_* is the tensor product of the Grassmann algebra generated by τ_0, τ_1, \dots and the polynomial algebra generated by ξ_1, ξ_2, \dots . ϕ_* is described by

$$\phi_*(\xi_k) = \sum_{i=0}^k \rho_{k-i} \beta^i \otimes \xi_i, \quad \phi_*(\tau_k) = \sum_{i=0}^k \xi_{k-i} \beta^i \otimes \tau_i + \tau_k \otimes 1.$$

Using this basis of \mathcal{S}_* , a dual basis is then given for the Steenrod algebra \mathcal{S}^* ; Q_k denotes the basis element dual to τ_k ; $\mathcal{P}^{r_1, \dots, r_s}$, dual to $\xi_1^{r_1} \xi_2^{r_2} \cdots \xi_s^{r_s}$, is a polynomial in the Steenrod operations. One has $\psi^*(Q_k) = Q_k \otimes 1 + 1 \otimes Q_k$, and more intricate formulas for $\psi^*(\mathcal{P}^{r_1, \dots, r_s})$.

The author defines [details are due to appear in a paper by Milnor and Moore] a conjugation c , which is a canonical automorphism of any connected Hopf algebra. An explicit formula for the conjugate in the dual algebra \mathcal{S}_* is given. It is applied to prove that the solution of the equation $\theta \mathcal{P}^1 = 0$, $\theta \in \mathcal{S}^*$, is given by the elements of the ideal $\mathcal{S}^* \mathcal{P}^{p-1}$. A last theorem proves that \mathcal{S}^* is a union of finite-dimensional subalgebras, which implies that every positive dimensional element of \mathcal{S}^* is nil-potent.

The author uses a special sign convention, by which interchanging objects of dimensions p and q introduces the sign $(-1)^{pq}$.

G. Hirsch (Brussels)

6093:

Negishi, Aiko. Exact sequences in the Steenrod algebra. J. Math. Soc. Japan 10 (1958), 71-78.

Il s'agit de l'algèbre de Steenrod $A^* = \sum_{n \geq 0} A^n$ pour $p=2$; elle est engendrée par les $\text{Sq}^n \in A^n$ reliés par les relations d'Adem. On note α_n l'endomorphisme de A^* défini par la multiplication à gauche par Sq^{2^n} , et M_n l'idéal à droite engendré par les Sq^{2^i} pour $i \leq n$ (on convient que $M_{-1} = M_{-2} = 0$). L'auteur explicite un système de générateurs pour les A^* -modules à droite A^*/M_n et M_n/M_{n-1} . L'application α_{n+1} induit, par passage aux quotients, des applications

$$A^*/M_{n-2} \xrightarrow{\alpha_{n+1}} A^*/M_{n-1} \xrightarrow{\alpha_{n+1}} A^*/M_n$$

qui forment une suite exacte pour $n \geq 0$ (les cas $n=0$ et $n=1$ avaient déjà été envisagés par Yamanoshita). La vérification nécessite des calculs laborieux.

H. Cartan (Paris)

6094:

Chang, Su-Cheng. On secondary products and spherical products. Acta Math. Sinica 6 (1956), 631-637. (Chinese. English summary)

In this note a relation is established between the secondary products which were introduced independently by the author [Acta Math. Sinica 3 (1953), 186-199; MR 17, 70] and J. M. Ebersold [Compositio Math. 12 (1954), 97-133; MR 16, 1044].

Sze-tsen Hu (Detroit, Mich.)

6095:

Shen, Sing-yao. Computations of secondary products. Acta Math. Sinica 8 (1958), 231-238. (Chinese. English summary)

Since the Whitehead products of homotopy groups satisfy the Jacobi identity, it is natural to ask whether there exist some similar relations between the secondary products introduced independently by S. C. Chang and J. M. Ebersold [cf. #6094 above]. A negative answer has been given in this paper.

Sze-tsen Hu (Detroit, Mich.)

6096:

Kodama, Y. On a closed mapping between ANR's. Fund. Math. 45 (1958), 217-227.

Various homotopy analogues of the Vietoris mapping theorem are proved. A typical example is the following theorem. If X and Y are finite dimensional ANR's (for metric spaces) and $f: X \rightarrow Y$ a closed map such that $f^{-1}(y)$ is an AR for each $y \in Y$, then f is a homotopy equivalence. A relative form of this theorem is also proved. These

results overlap considerably with those of the reviewer [Proc. Amer. Math. Soc. 8 (1957), 604-610; MR 19, 302].

S. Smale (Princeton, N.J.)

6097:

Wu, Wen-tsun. On the $\Phi_{(p)}$ -classes of a topological space. Sci. Record (N.S.) 1 (1957), 377-380.

In the present note the author generalizes his results on imbeddings of polyhedra into Euclidean spaces in his previous papers [Acta Math. Sinica 5 (1955), 505-552; 7 (1957), 79-101; Bull. Acad. Polon. Sci. Cl. III 4 (1956), 573-577; MR 17, 883; 20 #3536; 18, 664] to metrizable spaces. By means of Čech cohomology and the p -fold reduced product of a metrizable space X , the $\Phi_{(p)}$ -classes $\Phi_{(p)}^n(X)$ are defined. The least integer n , if it exists, for which $\Phi_{(p)}^n(X) \neq 0$ is called the $\Phi_{(p)}$ -index of X and will be denoted by $I_p(X)$. If there exists no such integer n , then $I_p(X)$ is defined to be ∞ . The integer p is always a prime. For the Euclidean N -space R^N or the N -simplex Δ^N , $N > 0$, we have $I_p(R^N) = N(p-1) = I_p(\Delta^N)$. It follows that if X is imbeddable in a Euclidean space R^N , then $I_p(X) \leq N(p-1)$, i.e., $\Phi_{(p)}^{N(p-1)}(X) = 0$. At the end of the note there are also some minor results.

Sze-Isen Hu (Detroit, Mich.)

6098:

Lee, Pei-shing. Characteristic classes on local coefficients. Sci. Record (N.S.) 1 (1957), 381-384.

In this note some methods of Pontrjagin and W.-T. Wu ["Sur les classes caractéristiques des structures fibrées sphériques", Actualités Sci. Ind., no. 1183 = Publ. Inst. Math. Univ. Strasbourg 11, pp. 5-89, 155-156, Hermann et Cie, Paris, 1952; MR 14, 1112] are generalized to local coefficients.

Sze-Isen Hu (Detroit, Mich.)

6099:

Koszul, J. L. Multiplicateurs et classes caractéristiques. Trans. Amer. Math. Soc. 89 (1958), 256-266.

Soit Y un espace, R une relation d'équivalence dans Y . Pour chaque entier $r \geq 0$, Y_r désigne l'espace des systèmes (y_0, \dots, y_r) de points équivalents; la collection des espaces Y_0, \dots, Y_r, \dots est munie de manière évidente d'une structure simpliciale, d'où un opérateur bord sur la somme directe des groupes de chaînes singulières $S(Y_r)$; alors $\sum_{p,r} S_p(Y_r)$ est un "complexe double"; soit $A(R, Y)$ le groupe différentiel gradué qu'il définit (muni du degré total et du bord total). $A(R, Y)$ s'envoie dans $S(Y/R)$, groupe des chaînes singulières du quotient Y/R , et plus exactement sur $\tilde{S}(Y/R)$, image de $S(Y) \rightarrow S(Y/R)$; ceci induit un isomorphisme des homologies $H_*(R, Y) \approx H_*(Y/R)$, en notant $H_*(R, Y)$ l'homologie de $A(R, Y)$, et $H_*(Y/R)$ celle de $\tilde{S}(Y/R)$. On a donc des homomorphismes

$$\mu_*: H_*(R, Y) \rightarrow H_*(Y/R) \text{ et } \mu^*: H^*(Y/R) \rightarrow H^*(R, Y),$$

qui sont des isomorphismes lorsque l'homologie de $S(Y/R)/\tilde{S}(Y/R)$ est nulle. C'est le cas lorsque tout simplexe singulier assez petit de Y/R est l'image d'un simplexe singulier de Y .

Soit maintenant G un group topologique opérant continûment (à droite) dans un espace Y ; soit R la relation d'équivalence définie par G , et Y/G le quotient. La collection des espaces $Y \times G^r$ ($r \geq 0$) est munie d'une structure simpliciale dont les opérations de face font intervenir la multiplication de G (comme dans le complexe standard servant à calculer l'homologie d'un groupe discret). La somme directe $\sum_{p,r} S_p(Y \times G^r)$ devient comme plus haut un complexe double, d'où un groupe différentiel gradué $B(G, Y)$, dont on note $H_n(G, Y)$ l'homologie. L'application continue $f_r: Y \times G^r \rightarrow Y_r$ qui envoie (y, g_1, \dots, g_r) dans

$(y, yg_1, yg_1g_2, \dots, yg_1 \dots g_r)$ est compatible avec les structures simpliciales, et induit une application $B(G, Y) \rightarrow A(R, Y)$; d'où $H_n(G, Y) \rightarrow H_n(R, Y)$ et $H^n(R, Y) \rightarrow H^n(G, Y)$, en notant $H_n(G, Y)$ et $H^n(G, Y)$ l'homologie et la cohomologie de $B(G, Y)$. Par composition avec μ_* et μ^* , on obtient

$$\chi_*: H_n(G, Y) \rightarrow H_n(Y/G), \quad \chi^*: H^n(Y/G) \rightarrow H^n(G, Y).$$

Théorème: si Y est fibré principal de groupe G , χ_* et χ^* sont des isomorphismes (en effet, les f_r sont des homéomorphismes). Ce théorème généralise des cas particuliers connus. Un cas important est celui où Y est réduit à un point P ; alors $B(G, P)$ joue le rôle d'espace classifiant pour le groupe G ; quant à $B(G, G)$ (G opérant dans G par translation), il est acyclique et joue le rôle de fibré universel. Pour tout fibré principal Y de groupe G , l'application naturelle $H^*(G, P) \rightarrow H^*(G, Y)$, composée avec χ^{*-1} , donne l'"homomorphisme caractéristique" $\gamma^*: H^*(G, P) \rightarrow H^*(Y/G)$, dont l'image se compose des "classes caractéristiques" de Y/G . Dans le cas où il existe un homomorphisme de Y dans un fibré universel, on obtient bien les classes caractéristiques au sens habituel.

L'Auteur applique cette théorie dans le cas où Y est fibré principal de base $X = Z/R$ et de groupe G , en supposant que Y soit "trivialisé" par $q: Z \rightarrow Z/R$, c'est-à-dire que le fibré de base Z induit par q soit trivial. Les classes de fibrés de base X , trivialisés par q , sont en correspondance bijective avec les "classes de multiplicateurs": un multiplicateur est une application continue $f: R \rightarrow G$ (R désignant le graphe de R dans $Z \times Z$) qui satisfait à la condition:

(M) Pour tout ouvert U assez petit de Z/R existe une application continue $g: q^{-1}(U) \rightarrow G$ telle que $f(z, z') = g^{-1}(z)g(z')$ chaque fois que $(z, z') \in R$ et $q(z) \in U$.

Deux multiplicateurs f_1 et f_2 sont équivalents s'il existe une application continue $h: Z \rightarrow G$ telle que $f_2(z, z') = h(z)^{-1}f_1(z, z')h(z')$ pour $(z, z') \in R$.

Étant donné un multiplicateur, f on l'utilise pour expliciter un homomorphisme $A(R, Z) \rightarrow B(G, P)$, d'où $H^*(G, P) \rightarrow H^*(R, Z)$. On montre que ceci est composé de $\gamma^*: H^*(G, P) \rightarrow H^*(X)$ (homomorphisme caractéristique du fibré Y défini par f) et de l'homomorphisme $\mu^*: H^*(X) \rightarrow H^*(R, Z)$. Lorsque μ^* est bijectif, on obtient ainsi un calcul de γ^* . Cette méthode est appliquée au cas où Z est fibré principal de base X , de groupe discret Γ , et où f est un multiplicateur sur Z à valeurs dans le groupe multiplicatif C^* des nombres complexes $\neq 0$: si Z est simplement connexe, on obtient un calcul de la classe de Chern du fibré Y_f défini par f , f étant écrit comme une application $Z \times \Gamma \rightarrow C^*$.

H. Cartan (Paris)

6100:

Halpern, Edward. The cohomology algebra of certain loop spaces. Proc. Amer. Math. Soc. 9 (1958), 808-817.

This paper contains a computation of the cohomology algebra of the space of loops on a space whose cohomology algebra is a truncated polynomial algebra, generated by an element of even dimension. Working initially in algebraic terms, the author proves that if (E_r) is an initially decomposable (split) spectral sequence of A -algebras (subject to the usual hypotheses), and $E_\infty^+ = 0$, and if $\sum E_2^{p,0} = B$ is a truncated polynomial algebra $A[x, h]$, where h is the height and x is the generator, $\deg x = m$, m even, $m \geq 2$, then $F = \sum E_2^{0,q} = \Lambda_A(Z_1) \otimes T_A(Z_0, Z_2, Z_4, \dots)$, where $Z_1 \in F$ is of degree $m-1$, and T_A is a twisted A -algebra of binomial type on generators Z_{2i} of degree

$i(hm-2)$. This latter means the free module on the (Z_m) with multiplication $Z_p Z_q = \binom{p}{q} Z_{p+q}$.

The proof is a direct application of standard methods in the theory of spectral sequences. The topological result follows by considering the space of paths with fixed base point on X , where X has cohomology a truncated polynomial algebra, generated by an element of even degree. The Serre spectral sequence then satisfies the conditions of the above theorem (taking coefficients in a principal ideal domain), and F is the cohomology algebra of ΩX .

D. W. Kahn (New Haven, Conn.)

6101:

Oniščik, A. L. On cohomologies of spaces of paths. Mat. Sb. N.S. 44(86) (1958), 3-52. (Russian)

Detailed account of results announced earlier [Dokl. Akad. Nauk SSSR 110 (1956), 932-935; MR 18, 919]. Chapter 1: Function spaces, the fiber space $P(X, A, x_0)$ of paths in X from x_0 to A , the loop space $\Omega(X)$, facts about spectral sequences. The characteristic map c^* of a fiber space $\mathcal{E} = (E, B, F, p)$ is obtained by composing the natural map from $H^*(F)$ to $H^*(P(E, F, x_0))$ with the isomorphism of the latter and $H^*(\Omega(B))$ under p^*-1 . Principal construction: to \mathcal{E} associate the fiber space ${}^1\mathcal{E}$, induced by $p: E \rightarrow B$ from the fibration ${}^1\mathcal{E} = (P(B, B, b_0) = P(B), B, \Omega(B))$; the total space of ${}^1\mathcal{E}$ can, for homotopy purposes, be identified with F ; the projection becomes the inclusion FCE . There is a natural map of ${}^1\mathcal{E} = (F, E, \Omega(B))$ into ${}^1\mathcal{H}$, and also, identifying $P(E)$ and $P(B)$ under p , a map of ${}^1\mathcal{H}$ into \mathcal{E} . Under suitable simple-connectedness conditions this can be iterated, giving rise to

$$\mathcal{E} \leftarrow {}^1\mathcal{H} \leftarrow {}^1\mathcal{E} \leftarrow {}^2\mathcal{H} \leftarrow {}^2\mathcal{E} \dots,$$

with ${}^2\mathcal{E} = (\Omega(B), F, \Omega(E))$ and ${}^3\mathcal{E} = (\Omega(E), \Omega(B), \Omega(F))$. The maps have a simple interpretation for the E_2 's of the spectral sequences. E.g., the map of the fibers in ${}^1\mathcal{H} \rightarrow \mathcal{E}$ induces the characteristic map c^* . Chapter 2: Some facts about Hopf spaces. All cohomology in characteristic 0. Theorem 2.1: Let $\pi_1(X) = 0$, and $H^*(X)$ be free anticommutative, generated by the subspace P ; let \bar{P} denote the transgressive in $H^*(\Omega(X))$. Then $H^*(\Omega(X))$ is free anticommutative generated by \bar{P} ; further, P maps by transgression onto \bar{P} . The proof is an algebraic argument with the spectral sequence, starting from the fact that $H^*(\Omega(X))$ is known to be free anticommutative. Theorem 2.2: Let (E, B, F) be a fiber space, E, B, F simply connected, $H^*(B)$ and $H^*(F)$ exterior algebras, $H^*(E)$ free. Then $H^*(E) = H^*(B) \otimes H^*(F)$. Proof: In the spectral sequence of ${}^3\mathcal{E}$ all degrees are even, the sequence is trivial, etc.

Theorem 2.3: Let $M = G/U$ be a simply connected homogeneous space of a semisimple compact Lie group; let the primitive elements of U [resp. G] be written as $P_1 \oplus P_2$ [resp. $Q_1 \oplus Q_2$] with $i^*: Q_1 \approx P_1$, $i^*(Q_2) = 0$. Then $H^*(\Omega(M)) = \Lambda P_2 \otimes K[Q_2^{(u)}]$; the upper index means that degrees are lowered by 1. The proof uses the map ${}^2\mathcal{E} \rightarrow {}^1\mathcal{H}$.

Chapter 3: Problem: "Given" $M = G/U$, to compute $H^*(G)$ from $H^*(U)$ or the reverse. This is solved by 2.3 if $H^*(\Omega(M_0))$ is known (M_0 = universal covering of M). Theorem 3.2: $U \sim 0 \Leftrightarrow H^*(M) = \text{exterior algebra} \Leftrightarrow H^*(\Omega(M)) = \text{polynomial algebra}$. Suppose from now on that $H^*(M)$ is the tensor product of algebras with one generator each.

Theorem 3.3 describes $H^*(\Omega(M))$, assuming $\pi_1(M) = 0$: Roughly speaking, for each generator of $H^*(M)$ of odd dimension $2s_i + 1$ one gets a polynomial factor of $H^*(\Omega(M))$, with generator of dimension $2s_i$; for each

generator of dimension $2l_j$ and height m_j one gets the tensor product of an exterior algebra with a generator of dimension $2l_j - 1$ and a polynomial algebra with a generator of dimension $2l_j m_j - 2$ [cf. the well-known case of the complex projective space; #6100 above]. The exterior part of $H^*(\Omega(M))$ is the c^* -image of $H^*(U)$. In particular the Poincaré polynomials satisfy

$$P_G(t) = P_U(t) \cdot \prod (1 + t^{2s_i+1}) \cdot \prod (1 + t^{2l_j m_j - 1}) \cdot (1 + t^{2l_j - 1})^{-1};$$

a similar relation holds for $i^*(H^*(G))$. The proof is a lengthy computation with spectral sequences. Theorem 3.4: The even generators of $H^*(M)$ generate the characteristic subalgebra C ; the positive-dimensional part of C is the kernel of p^* . The proof involves the map ${}^2\mathcal{E} \rightarrow {}^1\mathcal{H}$, algebraic lemmas about algebras of the type of $H^*(M)$, in particular about the special nature of the sub-algebra generated by the even generators, and H. Cartan's description of $H^*(M)$ in terms of the classifying spaces B_U, B_G , the map $\rho: B_U \rightarrow B_G$ and transgression from G to B_G [cf. A. Borel, Ann. of Math. (2) 57 (1953), 115-207; MR 14, 490].

H. Samelson (Ann Arbor, Mich.)

6102:

Boltjanskij, V. G. Homotopy classification of cross sections. Mat. Sb. N. S. 46(88) (1958), 91-124. (Russian)

This paper is primarily concerned with the homotopy classification problem relating to the second obstruction to a cross-section. The first five sections, however, are devoted to statements and proofs of known results on the first and second obstructions to extensions and homotopies (Hopf, Whitney, Pontryagin, Steenrod, Postnikov). The sixth and final section contains the statement and proof of the author's (second stage) homotopy classification theorem for cross-sections of fibre-bundles; this theorem effectively includes the known results on the second obstruction.

Let $\mathcal{P} = (P, p, B, C, G)$ be a fibre bundle whose fibre C is $(r-1)$ -connected and whose base B is an $(r+1)$ -dimensional polyhedron. Let σ_0 be a fixed cross-section of \mathcal{P} , let $c \in C$ be chosen as base-point and let $\Gamma \subseteq G$ be the stability subgroup of c ; we assume that Γ is path-connected. Then the cross-section σ_0 determines a fibre bundle $\tilde{\mathcal{P}} = (P, \tilde{p}, B, \Gamma, \Gamma)$ and there is a first obstruction $Y^2 \in H^2(B, \pi_r(\Gamma))$ to a cross-section of $\tilde{\mathcal{P}}$. The author also defines a pairing of $\pi_r(C)$ and $\pi_1(\Gamma)$ to $\pi_{r+1}(C)$ as follows: if $f: I^r \rightarrow C$, c represents α and $g: S^1 \rightarrow \Gamma$ represents β , then $\alpha\beta$ is represented by the map $(I^r \times I^2) \rightarrow C$, given by

$$h(x, y) = g(y), h(x), x \in I^r, y \in S^1, \\ h(x, y) = c, x \in I^r, y \in I^2.$$

Now let us suppose that σ, σ' are two cross-sections of \mathcal{P} which are homotopic on B^r ; there is then no real loss of generality in supposing that they coincide on B^r and agree with σ_0 on B^{r-1} . Then a difference cocycle $\partial^{r+1}(\sigma, \sigma')$ is defined whose class $D^{r+1} \in H^{r+1}(B, \pi_{r+1}(C))$ is the second obstruction to a homotopy between σ, σ' . Then the main theorem asserts that σ and σ' are homotopic if and only if there is an element $\Lambda^{r-1} \in H^{r-1}(B, \pi_r(C))$ such that

$$D^{r+1} = \Lambda^{r-1} \cup Y^2 + Sq^2 \Lambda^{r-1},$$

where the coefficient pairing for the cup-product is as above and Sq^2 is taken with respect to composition with the generator of $\pi_{r+1}(S^r)$. In this formula we need $r > 2$;

if $r=2$ a similar formula obtains, the role of the Steenrod square being taken over by the Postnikov square.

P. J. Hilton (Ithaca, N.Y.)

6103a:

Murasugi, Kunio. On the genus of the alternating knot. I, II. J. Math. Soc. Japan 10 (1958), 94-105, 235-248.

6103b:

Crowell, Richard. Genus of alternating link types. Ann. of Math. (2) 69 (1959), 258-275.

It is known [H. Seifert, Math. Ann. 110 (1935), 571-592] that the genus h of a knot k and the degree d of its polynomial always satisfy the inequality $d \leq 2h$. Crowell and Murasugi, working independently (on opposite sides of the earth), have each proved the quite unexpected result that equality holds for all alternating knots, and the not-so-deep but very amusing result that the Alexander polynomial $\Delta(x)$ of an alternating knot is alternating (i.e. the signs of the coefficients alternate).

Murasugi's proof of these assertions is to be found in his second paper, II. It is accomplished by direct argument on the Alexander matrix of the Dehn presentation of k . In his first paper, I, he proves that $d=2h$ for a special class of alternating knots. A knot belongs to this class if one of the surfaces constructed by the Reidemeister "checker-board" method from some alternating projection is orientable. The proof is rather easier for this case. The reviewer would like to remark that this special class of alternating knots is of some interest for itself. Although the proof of the general theorem in II makes reference to the proof of the more special result, it is logically independent of I.

Crowell's paper, the results of which were announced in Bull. Amer. Math. Soc. 62 (1956), 265, proves the basic results in a more general setting. Defining a reduced polynomial $\Delta(x)$ of a link with μ components to be the ordinary Alexander polynomial if $\mu=1$ and to be $(x-1)\Delta(x, \dots, x)$ if $\mu>1$, it is shown that the degree d of the reduced polynomial of an alternating link of genus μ is equal to $2h+\mu-1$ (that $d \leq 2h+\mu-1$ for any link, alternating or not, was shown by Torres [Ann. of Math. (2) 57 (1953), 57-89; MR 14, 574]), and that the reduced polynomial of an alternating link is alternating. The proof uses the Wirtinger projection and makes more formal use of graph theory, especially of the so-called matrix-tree theorem.

Both authors note that an orientable surface of minimal genus spanning a knot or link is always obtained when Seifert's method is applied to an alternating projection.

R. H. Fox (Princeton, N.J.)

6104:

Murasugi, Kunio. On the Alexander polynomial of the alternating knot. Osaka Math. J. 10 (1958), 181-189; errata, 11 (1959), 95.

The Alexander polynomial of an alternating knot of genus h is known to be of the form

$$\Delta(x) = \sum_{i=0}^{2h} (-1)^i a_i x^i,$$

where $a_i \geq 0$. [Cf. the preceding review.] It is now shown that $a_i > 0$ for each $i=0, \dots, 2h$. From this theorem it is deduced that almost all parallel knots, in particular, torus

knots of type p, q with $p > q > 2$ and cable knots (Schlauch-knoten) of higher order, are non-alternating.

R. H. Fox (Princeton, N.J.)

6105:

Crowell, Richard H. Nonalternating links. Illinois J. Math. 3 (1959), 101-120.

The existence of non-alternating knots (i.e., knots that have no alternating projection) was established by application of the theorem: The determinant D of an alternating link of μ components is not smaller than the minimal crossing number δ . This result was stated (for $\mu=1$) and incorrectly proved by Bankwitz [Math. Ann. 103 (1930), 145-161]. The author gives a complete demonstration, by a different method, and then proves several refinements of it, of which the easiest to state is the following: If a prime, alternating link is not the boundary of an unknotted Möbius band or circular ring then $2\delta \leq D+3$. With the aid of these refinements it is shown that the knots 8_{19} , 8_{20} , 8_{21} , 9_{42} , 9_{43} , 9_{44} , and 9_{46} are all nonalternating. Of the 84 tabulated prime knots there remain only eleven for which it is not known whether or not they are alternating.

R. H. Fox (Princeton, N.J.)

6106:

Zeidl, Bernhardine. Über 4- und 5-chrome Graphen. Monatsh. Math. 62 (1958), 212-218.

The first part of the paper is concerned with the structure of 4-chromatic graphs in relation to the complete 4-graphs. A graph is called node-critical if every one of its subgraphs having fewer nodes than the graph has a smaller chromatic number than the graph, and a graph is called critical if every one of its proper subgraphs has smaller chromatic number than the graph. A complete 4-graph is a graph with four nodes, every pair of distinct nodes joined by an edge. A graph obtained from a complete 4-graph by dividing edges into two through the insertion of new nodes any number of times is called a subdivision of a complete 4-graph. Such a subdivision of a complete 4-graph contains exactly four nodes of degree 3 and all its other nodes have degree 2; the four nodes of degree 3 may be called the corners of the subdivision.

The author proves the following theorem: Corresponding to each node of a critical 4-chromatic graph there is at least one subgraph of the graph which is a complete 4-graph or a subdivision of a complete 4-graph and of which the node is a corner.

This is an improvement on the theorem of the reviewer that every critical 4-chromatic graph contains as a subgraph a complete 4-graph or a subdivision of a complete 4-graph. The reviewer used only the following properties of critical 4-chromatic graphs: they are finite, they have no cut-node and the degree of every node is ≥ 3 . [J. London Math. Soc. 27 (1952), 85-92; MR 13, 572]. Here, in addition, the theorem of the reviewer that two nodes which are joined by an edge never form an isthmoid (cut-set) in a critical graph [Fund. Math. 40 (1953), 42-55; MR 15, 640] is used.

In the second part of the paper examples are given: 1. Of node-critical 4-chromatic graphs with $n=6m+4$ nodes, for all $m \geq 1$, in which the number of edges is $(n^2+5n)/6$; 2. of node-critical 5-chromatic graphs with $n=6m+5$ nodes, for all $m \geq 1$, in which the number of edges is $(n^2+9n-10)/6$. These graphs are not critical, however, and it remains uncertain how rich in edges critical 4-chromatic and 5-chromatic graphs can be.

G. A. Dirac (Hamburg)

DIFFERENTIAL GEOMETRY, MANIFOLDS

6107:

Ginatempo, Nicola. Sull'integrazione delle formule di Serret-Frenet. *Boll. Un. Mat. Ital.* (3) 13 (1958), 112-115.
 "An improvement is made on a result of T. Poeschl. The integration of the Serret-Frenet formulas is reduced to that of a differential linear homogeneous equation with variable coefficients. This equation is in a canonical form and intrinsic significations are given to the variable coefficients." (Author's summary)

J. De Cicco (Chicago, Ill.)

6108:

Vincensini, Paul. Sur certaines suites de points associées à une courbe gauche. *Rev. Fac. Sci. Univ. Istanbul Sér. A* 22 (1957), 31-33. (Turkish summary)

L. Biran avait démontré [même *Rev.* 21 (1956), 239-243 (1957); MR 19, 878] le théorème suivant: Soient $[P]$ une courbe gauche, lieu d'un point P , et P_1 le centre de sa sphère osculatrice en ce même point. Si la suite de points P, P_1, P_2, \dots, P_n , dont l'un quelconque à partir du second est le centre de la sphère osculatrice du lieu du précédent au même point, admet une position limite lorsque n augmente indéfiniment, cette position limite reste la même lorsque P décrit $[P]$ ou un arc de $[P]$. Le présent auteur en donne une démonstration cinématique qui montre de plus l'existence d'autres suites de points possédant la même propriété.

F. Sémin (Istanbul)

6109:

Franckx, Ed. Etude globale de la torsion des courbes qui sont tracées sur une surface. *Arch. Math.* 9 (1958), 378-381.

The author develops a theory for the torsions of curves analogous to the theory of Meusnier for the curvatures. Let F_2 be a family of curves on a surface having contact of order 2 at a regular point M_0 . Let τ be the torsion, θ the angle formed by the normal to the surface and the principal normal of the curves of F_2 , φ the angle formed by the principal normal and the line L which joins M_0 to the center of the osculating sphere. Then $k = \tau \sin(\theta - \varphi) / \cos \varphi$ is the same for all curves of F_2 . Unless the osculating sphere reduces to the osculating plane or is tangent to the surface, two curves of F_2 which have the same osculating sphere have the same torsion. Thus, spherical sections here play the same role as plane sections play in Meusnier's theory.

Let point T lie on the geodesic normal at a distance k from M_0 . Let Δ_T be the line through T parallel to the surface normal at M_0 . Let P be the intersection of Δ_T with the line L . Then M_0, T, P determine a circle, and the principal normal of the family F_2 cuts this circle at a point Q with $\tau = M_0Q$. This circle is analogous to the circle of Meusnier.

A. Schwartz (New York, N.Y.)

6110:

Bilinski, Stanko. A note on the fundamental equations of the theory of surfaces. *Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II.* 13 (1958), 121-124. (Serbo-Croatian summary)

6111:

Picone, Mauro. Il parametro monormale di una varietà regolare dello spazio euclideo. *Rev. Math. Pures Appl.* 1 (1956), no. 3, 9-14.

A slightly different version in *Atti Accad. Naz. Lincei.*

Rend. Cl. Fis. Mat. Nat. (8) 20 (1956), 705-711 has already been listed [MR 18, 758].

6112:

Blaschke, Wilhelm. Aus meiner geometrischen Werkstatt. *Rev. Math. Pures Appl.* 1 (1956), no. 3, 175-179.

6113:

Backes, F. Une propriété anallagmatique caractéristique des surfaces isothermiques. *Acad. Roy. Belg. Bull. Cl. Sci.* (5) 44 (1958), 94-100.

"L'obtention d'une propriété nouvelle des surfaces à courbure moyenne constante nous a suggéré son extension en géométrie anallagmatique: Toute surface isothermique est orthogonale à un cercle bien déterminé engendrant un système cyclique tel que, si l'on envisage la sphère tangente aux sphères principales aux points où celles-ci recoupent le dit cercle, les points caractéristiques de cette sphère sont inverses par rapport à la sphère harmonique."

Author's summary

6114:

Morduhai-Boltovskoi, D. D. Stereometric generalizations of Fagnano's theorem. *Rostov. Gos. Ped. Inst. Uč. Zap.* 4 (1957), 25-30. (Russian)

Consider a closed algebraic surface $f(x_1, x_2, x_3) = 0$ in E^3 , and on it algebraic curves whose coefficients are algebraic functions of the coefficients of f . Assume that the sum of the signed areas of the pieces of the surface bounded by such algebraic curves is again an algebraic expression in the coefficients of f . The paper shows that then the surface is generated, in a certain sense, by algebraic curves. The results are extended to surfaces representable in the form $x_1 = \varphi_1(u, v)$, where $\varphi_1(u, v)$ is a rational function of u and v .

H. Busemann (Cambridge, Mass.)

6115:

Nitsche, Johannes C. C. A uniqueness theorem of Bernstein's type for minimal surfaces in cylindrical coordinates. *J. Math. Mech.* 6 (1957), 859-864.

In a related paper [*Ann. of Math.* (2) 66 (1957), 543-544; MR 19, 878] the author has given a particularly simple demonstration of the theorem of S. Bernstein that a minimal surface $z(x, y)$ defined for all finite values of (x, y) is necessarily a plane. In the present work the author applies his method to prove that if a minimal surface admits a representation in cylindrical coordinates (r, φ, z) for all values of z and has a convex intersection with all planes $z = \text{const.}$, then the surface is a catenoid. It follows from a theorem of M. Schifman [ibid. 63 (1956), 77-90; MR 17, 632] that it is sufficient to assume the existence of an infinity of such planes corresponding to values of z which tend to plus and minus infinity. It should be of interest to determine to what extent this condition can further be weakened. R. Finn (Berlin)

6116:

Godeaux, Lucien. Sur une propriété de la surface focale commune de quatre congruences W . *Bull. Soc. Roy. Sci. Liège* 27 (1958), 217-220.

6117:

Marcus, F. Sopra i risultati di Fubini sull'inversione del teorema di permutabilità di Bianchi. *Boll. Un. Mat. Ital.* (3) 13 (1958), 189-195.

"Si dimostra in questo lavoro che i risultati di Fubini [*Ann. of Math.* (2) 41 (1940), 620-638; MR 2, 160] sul problema dell'inversione del teorema di permutabilità di

Bianchi, che gli parevano non risolvere completamente il problema, sono invece sufficienti a questo scopo, e possono condurre al medesimo risultato stabilito altrimenti da F. Marcus [Ann. of Math. (2) 49 (1948), 710-713; MR 10, 64].

Author's summary

6118:

Leruste, Philippe. Nombre de scalaires indépendants déterminés par des grandeurs tensorielles. C. R. Acad. Sci. Paris 248 (1959), 1121-1123.

6119:

Robinson, Lewis Bayard. Calculation of a complete system of tensors with the aid of symbolic multiplication. Math. Mag. 31 (1957/58), 5-14.

In dealing with the higher embedding theory in a projective curved n -dimensional space the reviewer derived the complete set of tensors associated with the set of partial differential equations

$$(1) \quad \partial_{a_1 \dots a_n} x = \psi_{a_1 \dots a_n} x + \sum_{r=1}^n \psi_{a_1 \dots a_{n-1} r} b_r \partial_{b_1 \dots b_n} x$$

defining the subspace under consideration [Nederl. Akad. Wetensch. Proc. 53 (1950), 318-326, 487-493, 835-847, 848-856; MR 12, 282]. The author of the present paper confines himself to the case $n=N=1$ and finds by another method the complete set of quadratic tensors. He generalizes Wilczynski's method of infinitesimal Lie transformations and uses substantially the Clebsch-Aronhold symbolism. In this way he obtains a set S of infinitesimal transformations of the quadratic tensors under consideration. Solving S he obtains a functionally complete system of tensors associate with the system (1) for $n=N=1$. For details of the rather involved computation the reader must be referred to the original paper.

V. Hlavatý (Bloomington, Ind.)

6120:

Villa, Mario. Dall'applicabilità delle superficie a quella delle trasformazioni. Rend. Sem. Mat. Fis. Milano 27 (1957), 100-109.

"Metric and projective applicability of surfaces in three dimensions are first shortly treated; recent extensions about surfaces in space with more than three dimensions are also touched upon. The recent notion of projective applicability of two point transformations between planes is then treated under analytic and geometric respect. Finally several researches suggested by the new ideas thus introduced are announced." (Author's summary)

J. De Cicco (Chicago, Ill.)

6121:

Cantoni, Lionello. Le trasformazioni puntuali fra due spazi proiettivi in una coppia a direzioni caratteristiche indeterminate. Boll. Un. Mat. Ital. (3) 13 (1958), 79-83.

Preliminary announcement of results on a point-transformation between two projective n -dimensional spaces in a pair of corresponding points where the transformation is regular and the characteristic directions are undetermined (i.e., to each inflexional element through the point in one space corresponds an inflexional element in the other space).

E. Bompiani (Rome)

6122:

Kaul, R. N. Union curvature of a vector field. Tensor (N.S.) 7 (1957), 185-189.

A union curve of a hypersurface V_n immersed in a Riemannian space V_{n+1} has the property that its oscu-

lating geodesic surface at each point of the curve contains that line of a congruence of curves which passes through the point. The union curvature of a curve has been defined analogously to geodesic curvature, but so that the union curvature of a union curve is zero. [C. E. Springer, Bull. Amer. Math. Soc. 51 (1945), 686-691; MR 7, 172.] The present paper considers the union curvature of a vector field with respect to a curve on a hypersurface V_n of V_{n+1} . This includes the union curvature of a curve as a special case when the vector defined for each point of the curve is tangent to it. Simple geometric consequences of the definition are obtained.

A. Fialkow (Brooklyn, N.Y.)

6123:

Prvanović, Mileva. Système des courbes cycliques d'un sous-espace plongé dans un espace riemannien. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske Ser. II. 12 (1957), 233-243. (Serbo-Croatian summary)

Blaschke [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (6) 2 (1925), 399-400] has defined a cyclic family of curves on a surface in euclidean 3-space as one for which the osculating circles of the curves of the family passing through each point P intersect again a circle which passes through P orthogonal to the surface. The author generalizes this definition to apply to curves on any subspace of an m -dimensional Riemann space and derives their differential equations. These same equations have been studied by the reviewer previously in another connection [Trans. Amer. Math. Soc. 45 (1939), 443-473; p. 446]. It is shown that families of conformal geodesics and families of union curves are types of cyclic curves. Some conformal properties of cyclic curves are given.

A. Fialkow (Brooklyn, N.Y.)

6124:

Boboc, N. Sur la caractérisation des variétés différentiables à base dénombrable. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 391-395.

Al noto teorema che ogni varietà a base numerabile ammette una metrica riemanniana, si aggiunge il teorema reciproco che ogni varietà metrizzabile ha una base numerabile. La dimostrazione di quest'ultimo teorema dipende da un lemma di Urysohn (citato incorrettamente nel testo), che afferma che ogni spazio connesso, metrico e localmente compatto ha base numerabile. L'Autore preferisce alla dimostrazione classica del teorema un'altra, basata sulla teoria del potenziale, supponendo che la metrica sia riemanniana e sufficientemente regolare (il che non è necessario per la dimostrazione topologica). Pertanto la dimostrazione dell'Autore non sembra offrire alcun vantaggio su quella classica, tranne l'analogia con la dimostrazione del teorema di T. Radó, secondo cui ogni superficie di Riemann (senza supporre l'esistenza di una metrica) ha base numerabile.

E. Calabi (Princeton, N.J.)

6125:

Yano, Kentaro. Sur les vecteurs harmoniques et vecteurs de Killing dans un espace de Riemann à frontière. C. R. Acad. Sci. Paris 247 (1958), 1085-1087.

In this note the author announces the following results. Let M be a compact orientable Riemannian manifold with boundary B . If the Ricci curvature of M is positive definite [negative] and the second fundamental form of B is non-positive [non-negative], then there does not exist a harmonic [Killing] non-zero vector field which is tangent on the boundary. Furthermore, if the Ricci curvature of M is positive definite [negative] and the mean curvature of B is non-positive [non-negative], then there does not

exist a harmonic [Killing] non-zero vector field which is normal on the boundary.

The author also announces the following necessary and sufficient conditions that a vector v^μ should be harmonic [Killing]:

$$g^{\mu\lambda}\nabla_\mu\nabla_\lambda v^\mu - K_\lambda v^\lambda = 0$$

$$[g^{\mu\lambda}\nabla_\mu\nabla_\lambda v^\mu + K_\lambda v^\lambda = 0, \nabla_\lambda v^\lambda = 0] \text{ on } M;$$

and

$$\alpha(\nabla_\lambda v^\lambda) - 2(\nabla_{(\mu} v_{\lambda)})v^{\mu\lambda} = 0 \quad [(\nabla_\mu v_\lambda)v^{\mu\lambda} = 0] \text{ on } B.$$

J. J. Kohn (Waltham, Mass.)

6126:

Mokanu, P. [Mocanu, P.] *Espaces partiellement projectifs*. Z. Cist. Prikl. Mat. 1 (1956), 75-108. (Russian)
A translation from the Romanian in Acad. R.P. Romine. Stud. Cerc. Mat. 6 (1955), 495-528 [MR 18, 67].

6127:

Bușmanova, G. V. *Weyl and Riemannian geometries induced on a surface by the straight lines of a canonical pencil*. Kazan. Gos. Univ. Uč. Zap. 112 (1952), no. 10, 109-115. (Russian)

Using the concepts developed by A. P. Norden [Trudy Sem. Vektor. Tenzor. Analizu 6 (1948), 125-224; MR 15, 61], and by herself [Kazan. Gos. Univ. Uč. Zap. 110 (1952), no. 3], the author develops the interior geometries of a surface in the case of Weyl or Riemann with the aid of the lines of a canonical pencil for normals of the first and second kind: both the point and plane points of view are taken into consideration. Among the special cases investigated is the case that the lines of the canonical pencil induce on the surface a Riemannian geometry with the line element $ds^2 = du^2 + 2 \cos t \, dudv + dv^2$, $\tau = \tau(u+v)$; the case of isothermic-asymptotic surfaces; and that of constant curvature. D. J. Struik (Cambridge, Mass.)

6128:

Rozenfel'd, B. A. *A geometric interpretation of symmetric spaces with simple fundamental groups*. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 23-26. (Russian)

Non-euclidean spaces are introduced as projective spaces under algebras of real and complex numbers, quaternions and octonions, in which a distance ω between points is introduced. For example, in the case of the complex projective space

$$\cos^2 \omega = (\sum \bar{y}^i y^i)^{-1} (\sum \bar{x}^i x^i) (\sum \bar{x}^i x^i)^{-1} (\sum \bar{y}^i y^i),$$

where the bar indicates complex conjugacy.

A geometric interpretation is given of the symmetric spaces of E. Cartan, possessing simple fundamental groups, in the form of manifolds of images of symmetry in the indicated non-euclidean spaces. A table is given, containing simple Lie groups (both compact and non-compact), non-euclidean spaces with representations of these groups, stationary subgroups of symmetric spaces and representations corresponding to their symmetry in the space. G. I. Kručovič (RŽ Mat 1957 #7324)

6129:

Hwang, Cheng-chung. *On the isometric correspondence of Riemannian spaces of category $n-2$* . Acta Math. Sinica 8 (1958), 222-230. (Chinese. English summary)

Let $R(n, p)$ denote a regular n -dimensional Riemannian space of category p , i.e., a Riemannian space which admits p and only p functionally independent invariants. The problem to determine for a pair of such spaces a set

of scalar relationships which imply that the spaces are isometric has been solved for $p=n$ and for $n=2$, $p=2$, 1, 0 by T. Y. Thomas [Proc. Nat. Acad. Sci. U.S.A. 31 (1945), 306-310; MR 7, 80]. The author supplies in a restricted sense an answer for $p=n-2$ according as each of the spaces admits as its complete group of motions (1) a 3-dimensional local Lie group, or (2) an abelian local group of motions G_2 , or (3) a non-abelian local group of motions G_2 . T. K. Pan (Norman, Okla.)

6130:

Vranceanu, G. *Sur la représentation géodésique des espaces de Riemann*. Rev. Math. Pures Appl. 1 (1956), no. 3, 147-165.

Two Riemannian spaces V_n and \bar{V}_n are in geodesic correspondence if

$$\Gamma_{jk}^i = \bar{\Gamma}_{jk}^i + \delta_j^i \psi_k + \delta_k^i \psi_j, \quad \psi_i = \partial_i \psi \quad (i=1, \dots, n).$$

It is first shown that if \bar{V}_n is an Einstein space (here called space of constant curvature of the second kind, $R_{ij} = \beta g_{ij}$, β constant) and V_n is symmetrical, both V_n and \bar{V}_n are of constant curvature. In the general case the orthogonal congruences which V_n and \bar{V}_n have in common are introduced, and with them the concept of V_n of category m ; for such a V_n the components of ψ_i with respect to these congruences satisfy the conditions $\psi_\alpha = 0$ ($\alpha, \beta = m+1, \dots, n$), $\psi_k \neq 0$ ($k=1, \dots, m$). Then, if the correspondence between V_n and \bar{V}_n is not trivial the case $m=n$ leads to the representation of the ds^2 and $d\bar{s}^2$ in the form of Levi-Civita; if $m < n$ the ds^2 can be reduced to the form

$$ds^2 = a_i \phi'(x^i) (dx^i)^2 + \phi(1) c_{\alpha\beta} dx^\alpha dx^\beta,$$

where $\phi(x) = (x-x^1) \cdots (x-x^m)$, the a_i depend on x^i only and the $c_{\alpha\beta}$ do not depend on the x^i . Now the theorem of Sinyukov [Dokl. Akad. Nauk SSSR 98 (1954), 21-23; MR 16, 515] is demonstrated for V_n of category m , $1 < m \leq n$. It is also shown that a symmetrical irreducible V_n , not of constant curvature, is completely determined by its geodesics; also, a space V_n of category 0 is determined by its geodesics but for a multiplicative constant. D. J. Struik (Cambridge, Mass.)

6131:

Vranceanu, G. *Sur les espaces A_3 non projectivement euclidiens à groupe transitif \mathcal{G}_7* . J. Math. Soc. Japan 10 (1958), 221-233.

In an affine space A_3 , which is not projectively euclidean, which has a stability group G_4 (and hence admits a group of motions G_4), the contracted curvature tensor $p_{kl} = \gamma_{khl}^h = p_{lk}$ has rank ≤ 1 . Then the curvature tensor can be written in such a way that all its components are zero except $\gamma_{223}^1 = 1$, γ_{212}^1 , γ_{221}^2 . The example $\Gamma_{22}^1 = x^3 - \alpha x^1$, $\Gamma_{22}^2 = \beta x^1$ (α, β constants), all other Γ zero,

$$x'^1 = r x^1 + s x^3 + p, \quad x'^2 = x^2 + b,$$

$$x'^3 = \beta s x^1 + (r + \alpha s) x^3 + p'' + \alpha p,$$

$p = p(x^2)$ satisfying $p^{1v} + \alpha p'' - \beta p = 0$ and r, s, b constants, shows the existence of such A_3 ; the group contains indeed seven constants (r, s, b and the four in p). If we consider A_n for which $\Gamma_{22}^1 = x^3 - \alpha x^1$, $\Gamma_{22}^2 = \beta x^1$ and all others zero, then their group of motions contains $n^2 - 3n + 7$ parameters if $\beta = 0$ and $n^2 - 4n + 10$ parameters if $\beta \neq 0$. A preface gives a short survey of the many results reached during the last ten years in the determination of the number of parameters of the motion groups in affine and projective spaces.

D. J. Struik (Cambridge, Mass.)

6132:

Cossu, Aldo. Sui movimenti quasi-affini in una varietà a connessione affine. *Rend. Mat. e Appl.* (5) 16 (1957), 58-73.

The author studies, for an affine space A_n with parameters $L_{jh}^i(x^k)$, a transformation (1) of class C^u defined in a domain of local coordinates $y^i = y^i(x^j)$ with non-vanishing Jacobian, called an affine motion in the wider sense [K. Yano, *Gruppi di trasformazioni in spazi geometrici differenziali*, Istituto Matematico, Roma, 1953-54; MR 17, 404; p. 85], distinguishing it from the affine motion in the weak sense [J. A. Schouten, *Ricci calculus*, Springer, Berlin-Göttingen-Heidelberg, 1954; MR 16, 521; p. 346], and introducing the term quasi-affine motion. The results are the following. I. In order that (1) be a quasi-affine motion, it is necessary and sufficient that $H_{hh'}^i(y^k) - H_{hh'}^i(x^k) = 0$ ($H_{hh'}^i = L_{hh'}^i - \delta_{hh'}^i L_{hh}^p/n$, $i = i', h = h', h' = h'$) holds. II. Every quasi-affine motion in A_n with parameters L_{jh}^i is also a quasi-affine motion for the T_p -transform of A_n . Here T_p -transformation means $L_{jh}^i = L_{jh}^i + \delta_{jh}^i \varphi_h$ [cf. E. Bortolotti, *Ann. di Mat.* (4) 8 (1930), 53-101; p. 76]. III. The necessary and sufficient condition that a quasi-affine motion for the A_n with L_{jh}^i be a quasi-affine motion for the T_p -transform of A_n is that φ_h satisfies $\varphi_h(y^k) + (2/(n-1))\Phi_h(y^k) = \varphi_h(x^k) + (2/(n-1))\Phi_h(x^k)$, where $\Phi_h = L_{[hh]p}$ is the Einstein vector (in the nomenclature of Bortolotti). IV. Every quasi-affine motion for A_n with L_{jh}^i is an affine motion for the T_p -transform with $P_{hh'}^i = L_{hh'}^i - (2/(n-1))\delta_{hh'}^i \Phi_h$. V. The only A_n with asymmetric affine connection for which every quasi-affine motion is an affine motion are those with $\Phi_h = 0$. VI. The necessary and sufficient condition that an infinitesimal transformation (2) $y^i = x^i + \xi^i dt$ be a quasi-affine transformation for the A_n is that the corresponding Lie derivative $XH_{hh'}^i$ vanishes. VII. The necessary and sufficient condition that an infinitesimal transformation be a weak quasi-affine one for the A_n is that in the tensor formation $T_{hh'}^k \dots \nabla_{\xi} X \nabla_{\xi} T_{hh'}^k - T_{hh'}^k \dots \nabla_{\xi} X \nabla_{\xi} T_{hh'}^k$, the Lie derivative and the covariant derivative are permutable. VIII. In order that the infinitesimal transformation (2) be a weak quasi-affine one for A_n , it is necessary and sufficient that the Lie derivative and the covariant derivative of the mixed tensors with equal numbers of the indices of covariancy and contravariancy are permutable. IX. A weak quasi-affine transformation for the A_n with L_{jh}^i is a weak affine one for the T_p -transform when and only when $X\varphi_h = (2/(n-1))X\Phi_h$. X. A weak quasi-affine transformation for a A_n is a weak affine one when $X\Phi_h = 0$. XI. In order that weak affine transformation (2) be a quasi-affine one for an A_n which subordinates an absolute parallel displacement of directions, it is necessary and sufficient that, for the Lie derivative of the vectors of a complete system, (3) $XX_{\alpha}^i = (\varepsilon_{\alpha}^{\beta} + \delta_{\alpha}^{\beta} \sigma)X_{\beta}^i$ holds, where σ is a scalar function and $\varepsilon_{\alpha}^{\beta}$ are constants. XII. In order that (2) be a weak quasi-affine transformation for the A_n which determines an absolute displacement of vectors, it is necessary and sufficient that for the Lie derivatives of n independent and absolutely parallel vectors, (3) is satisfied. XIII. When, for an A_n which determines an absolute parallel displacement of directions, (2) is a weak quasi-affine transformation, there exists among the T_p -transforms of the A_n a manifold of zero curvature, for which (2) is a weak affine transformation. XIV. When an A_n admits a weak quasi-affine transformation, it admits a group of quasi-affine motions with parameters formed from the weak quasi-affine one.

XV. The necessary and sufficient condition that an A_n admit a one-parametric group of quasi-affine motions is that there exist a system of coordinates in which the $H_{hh'}^i$ are homogeneous functions of degree -1 in the same coordinates. XVI. A (complicated) necessary and sufficient condition is given that an A_n with L_{jh}^i admit a group of quasi-affine motions. XVII. The manifolds with an affine connection admitting a group of maximum order $n^2 + n$ of quasi-affine motions are those with semi-symmetric connections as well as absolute displacements of directions.

T. Takasu (Yokohama)

6133:

Cossu, A. Movimenti in una varietà a connessione tensoriale. *Rend. Mat. e Appl.* (5) 16 (1957), 118-130.

E. Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1 (1946), 478-482, 483-485; MR 8, 404] defined in a differentiable manifold A_n of class C^u a tensorial connection for tensors λ^{ik} with double contravariant indices, assigning for every domain of local coordinates n^5 functions $L_{rst}^{ik}(x^j)$ which obey the transformation law $L_{rst}^{ik}(x^j) = L_{rst}^{ik}(y^j) \partial y^r / \partial x^r \partial y^s / \partial x^s \partial y^t / \partial x^t \partial x^k / \partial y^k - \partial y^i / \partial x^i \partial y^k / \partial x^k \partial y^r / \partial x^r \partial y^s / \partial x^s \partial y^t / \partial x^t \partial x^k / \partial y^k$. The covariant differential was $\nabla \lambda^{ik} = d\lambda^{ik} + L_{rst}^{ik} \lambda^{rs} dx^t$. In this note a notion of motion is introduced which is analogous to the affine motion of the paper reviewed above [6132], and results I-XI below are obtained, which are analogous to the corresponding ones among I-XVII listed in that review. In particular, the author finds, by reasoning analogous to that of I. P. Egorov [Dokl. Akad. Nauk SSSR 64 (1949), 621-624; MR 10, 739], the maximum order of groups of motions, which a particular manifold with tensorial connection can subordinate.

I. The necessary and sufficient condition that the infinitesimal transformation (1) $y^i = x^i + \xi^i dt$ be a motion for the A_n with L_{rst}^{ik} is that the Lie derivative of the connection parameters with respect to (1) vanishes. II. In order that the transformation (1) be an infinitesimal motion for the A_n with L_{rst}^{ik} , it is necessary and sufficient that the Lie derivatives of λ^{ik} with respect to (1) and the covariant derivative are commutative. III. In order that an A_n with L_{rst}^{ik} admit an infinitesimal motion, it is necessary and sufficient that a coordinate system exist in which the connection parameters are homogeneous functions of degree -1 in the coordinates. IV. Every infinitesimal motion for an A_n with L_{rst}^{ik} is an infinitesimal motion for the manifold with $\Gamma_{rst}^{ik} = \frac{1}{2}(L_{rst}^{ik} + L_{rst}^{ik})$. V. Every infinitesimal connection with L_{rst}^{ik} is an infinitesimal affine motion for the two manifolds with affine connections of parameters $P_{rst}^i = \frac{1}{2}L_{rst}^{ip}$ and $Q_{rst}^i = \frac{1}{2}L_{rst}^{ip}$ respectively. VI. The necessary and sufficient condition that, for an A_n with $L_{rst}^{ik} = \delta_{rs}^i M_{tr}^k + \delta_r^i N_{ts}^k$ [A. Cossu, *Rend. Mat. e Appl.* (5) 15 (1956), 190-210; MR 18, 599; see p. 199], (1) be an infinitesimal motion is that $XL_{tr}^i = 0$ and $X(R_{tr}^i - \delta_r^i R_{tp}^p/n) = 0$ hold, where $L_{tr}^i = \frac{1}{2}(M_{tr}^i + N_{tr}^i)$, $R_{tr}^i = \frac{1}{2}(M_{tr}^i - N_{tr}^i)$. VII. When an A_n with L_{rst}^{ik} admits an infinitesimal motion, it admits a one-parametric group of motions generated from the infinitesimal motion. VIII. The necessary and sufficient condition for an A_n with L_{rst}^{ik} to admit a one-parametric motion is that there exist a system of coordinates x^i in which the L_{rst}^{ik} are homogeneous functions of degree -1 in x^i . IX. A (complicated) necessary and sufficient condition is given for an A_n with L_{rst}^{ik} to admit a group of motions. X. The order of a group of motions in an A_n with L_{rst}^{ik} , which is not deduced from a vectorial connection, is always inferior to $n + n^2$. XI. The maximum

order of a group of motions in an A_n with $L_{rkt} = \delta_k M_{rt} + \delta_r N_{tk}$, which is not deduced from a vectorial connection, is n^2 .

T. Takasu (Yokohama)

6134:

Cossu, Aldo. Movimenti in una varietà dotata di una connessione per tensori doppi misti. Rend. Mat. e Appl. (5) 16 (1957), 454-467.

In this note the notion of motion is introduced in a tensorial connection, for doubly mixed tensors λ_k^i [A. Cossu, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19 (1955), 421-427; MR 18, 233], which obey the law of transformation $E_{r'k'}^{i'} = E_{rkt}^{i'} \partial_r \partial_k \partial_t \partial_{t'}$ — $\partial_{r'} \partial_{k'} \partial_{t'} \partial_t = \partial_{k'} \partial_{r'} \partial_t \partial_t$ for the local coordinate transformation $x^i = x^{i'}(x')$, the covariant differential being $d\lambda_k^i + E_{rkt}^{i'} \lambda_{s'}^i dx^s$. The results are mostly analogous to those in the case of the tensors λ_k^i [6133 above]. The exposition is limited to those which are quite new. I. The necessary and sufficient condition that $y^i = f^i(x)$ be a motion of the manifold with the tensorial connection $E_{rkt}^{i'}$ is that $E_{rkt}^{i'}(y^j) - E_{r'k't'}^{i'}(x^j) = 0$ ($i = i', r = r', \dots$) holds. II. In order that (1) $y^i = x^i + \xi^i dt$ be an infinitesimal motion in the manifold with $E_{rkt}^{i'}$, it is necessary and sufficient that the Lie derivative $XE_{rkt}^{i'} = 0$. III. The necessary and sufficient condition that (1) be an infinitesimal motion is that the covariant derivative and the Lie derivative of λ_k^i with respect to (1) are permutable. IV. The necessary and sufficient condition that a manifold with $E_{rkt}^{i'}$, which subordinates an absolute displacement of tensors, admit (1) as infinitesimal motion, is that the Lie derivatives of linearly independent and absolutely parallel λ_k^i with respect to (1) are linear combinations of them [for an affine analogue, cf. H. P. Robertson, Ann. of Math. (2) 33 (1932), 496-524]. V. Every infinitesimal motion for a manifold with (2) $E_{rkt}^{i'} = \delta_k^i M_{rt} - \delta_r^i N_{tk}$ [A. Cossu, Rend. Mat. e Appl. (5) 15 (1956), 190-210; MR 18, 599] is a quasi-affine infinitesimal motion for the manifold with vectorial connection of parameter $G_{tr}^i = \frac{1}{2}(M_{tr}^i + N_{tr}^i)$. VI. Every infinitesimal motion for the manifold of parameters (2) is an affine motion for the manifold with vectorial connection of the parameters $P_{tr}^i = G_{tr}^i - 2\delta_r^i G_{(tj)p}^j/(n-1)$. VII. The necessary and sufficient condition that (1) be an infinitesimal motion for the manifold with the parameters (2), is that $XT_{tr}^i = 0$ and $XP_{tr}^i = 0$ are satisfied, where $T_{tr}^i = \frac{1}{2}(M_{tr}^i - N_{tr}^i)$. VIII. When the manifold with $E_{rkt}^{i'}$ admits an infinitesimal motion, it also admits a one-parametric group of motions generated from the infinitesimal ones. IX. The necessary and sufficient condition that the manifold with $E_{rkt}^{i'}$ admit a one-parametric group of motions, is that there exists a system of coordinates x^i in which the parameters $E_{rkt}^{i'}(x^j)$ are homogeneous functions of degree -1 in x^j . X. A (complicated) necessary and sufficient condition is given that a manifold with $E_{rkt}^{i'}$ admit a group of motions. XI. The necessary and sufficient condition that a manifold with $E_{rkt}^{i'}$, which subordinates an absolute displacement, admit a one-parametric group of special motions, is that there exists a system of coordinates in which the linearly independent and absolutely parallel tensors λ_k^i are independent of one and the same coordinate. XII. In order that a manifold with $E_{rkt}^{i'}$, which subordinates an absolute displacement, admit of a one-parametric group of general motions, it is necessary and sufficient that there exists a system of coordinates in which linearly independent and absolutely parallel tensors $\lambda_{\alpha k}^{\beta i}$ are of the form $\lambda_{\alpha k}^{\beta i} = \mu_{\alpha\beta}^i(x^1) \nu_{\alpha k}^{\beta i}(x^2, \dots, x^n)$ with the condition

$d\mu_{\alpha\beta}^i/dx^1 = c_{\alpha\beta}^i \mu_{\alpha\beta}^i$, where $c_{\alpha\beta}^i$ are constants. Two other results, XIII and XIV, are given, which are rather complicated.

T. Takasu (Yokohama)

6135:

Walker, A. G. Connexions for parallel distributions in the large. II. Quart. J. Math. Oxford Ser. (2) 9 (1958), 221-231.

In a previous paper [same J. 6 (1955), 301-308; MR 19, 312] the author showed that on a differentiable manifold an affine connection always exists globally with respect to which one or more given distributions [cf., e.g., C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, 1946; MR 7, 412; p. 86] are parallel. It was also shown that, if the given system of distributions is integrable, the connection can be chosen to be symmetric. The present paper continues the study of global connections related to given distributions, and relative parallelism and path-parallelism with respect to a connection are defined and considered. A number of existence theorems below for global connections are given, these connections being torsion-free whenever possible, and in each case a formula for the simplest connection having the desired properties is constructed. All such formulae are expressed in terms of the projection tensors associated with given distributions, and are simplified by means of a convenient notation for the various projections of a tensor. With this notation the calculations are very similar to those resulting from the use of forms and special frames. The present method has the advantage, however, that the resulting formulae for tensors and connections are expressed in relation to a general coordinate system: no transformation from a special to a general frame is necessary, and formulae are in a convenient form for subsequent applications to special problems. It is generally assumed that the given structure (manifolds and distributions) is of class C^∞ . If, however, the given structure is analytic and if the manifold is such that there exists globally an analytic connection, then the constructed connections will be analytic. The main results are as follows. Theorem 1: For any complementary distributions D', D'' there is a global symmetric connection with respect to which both distributions are path-parallel. Theorem 2: For any complementary distributions D', D'' there is a global symmetric connection with respect to which each distribution is path-parallel relative to the other. Theorem 3: For any complementary distributions D', D'' there is a global connection with respect to which D' is parallel and D'' is path-parallel and parallel relative to D' ; this connection can be chosen formally so that it becomes symmetric when D' is integrable. Theorem 4: For any complementary distributions D', D'' which are orthogonal with respect to a metric g , there is a global connection L with respect to which D' and D'' are parallel and g is constant, i.e., $g_{ij;k} = 0$. A remark is added: "In a later paper it will be shown how some of the present results lead to a definition of 'torsional derivation' and the construction of new concomitants for an almost complex structure. Certain holonomic properties of some of the connections given here will also be discussed in another paper, where it will be shown how they are related to the foliation (lamination) groups of Ehresmann." T. Takasu (Yokohama)

6136:

Takasu, Tsurusaburo. Generalized Riemannian geometry. I. Yokohama Math. J. 5 (1957), 115-169.

The author's "generalized Riemannian space" is one in which the element of distance ds is given by

$$ds^2 = g_{\mu\nu} dx^\lambda dx^\mu dx^\nu \quad (\mu, \nu, \lambda = 1, \dots, n).$$

A symmetric linear connection, depending on position only, is defined in terms of derivatives of $g_{\mu\nu}$ such that the corresponding covariant derivative of the $g_{\mu\nu}$ vanishes identically. [See also Liber, *Trudy Sem. Vektor. Tenzor. Analizu* 9 (1952), 319-350; MR 14, 688; and Wegener, *Nederl. Akad. Wetensch. Proc. Ser. A* 38 (1935), 949-955.] A curvature tensor is defined by means of the conditions of integrability of the parallel displacement. This gives rise to two distinct tensors, according as indices are raised or lowered by means of $g_{\mu\nu}$ or by the metric tensor $g_{\mu\alpha}$ of the corresponding Finsler space defined by

$$F^2(x, dx) = g_{\mu\nu} dx^\lambda dx^\mu dx^\nu.$$

The identities satisfied by these curvature tensors are derived, and analogues of the Bianchi identities and Schur's theorem are obtained. {At some stages of the argument the reviewer was unable to follow the author's reasoning; in particular, he cannot agree with equation (11.4), which gives the relation between $g_{\mu\alpha}$ and $g_{\mu\nu}$.}

H. Rund (Durban)

6137:

Laptev, G. F. A hypersurface in the space of projective connection. *Dokl. Akad. Nauk SSSR* 121 (1958), 41-44. (Russian)

L'A. donne les généralisations des formes asymptotiques et de Darboux, d'élément linéaire projectif de Fubini, des quadriques de Darboux et de Lie et des faisceaux des droites canoniques de la surface plongée dans l'espace projectif au cas d'une hypersurface plongée dans l'espace à connexion projective. Il construit ces notions et l'objet fondamental complet par ses méthodes [Trudy Moskov. Mat. Obšč. 2 (1953), 275-382; MR 15, 254].

A. Švec (Prague)

6138:

Sasaki, Shigeo. The minimum number of points of inflexion of closed curves in the projective plane. *Tôhoku Math. J.* (2) 9 (1957), 113-117.

Let Γ be a "smooth" closed curve in a projective plane which is not homotopic to zero ("smoothness" is not explicitly defined but the author uses twice continuous differentiability). Theorem 1: Γ has inflection points. Theorem 2: If Γ is simple, it has at least three inflection points. The proofs are metric.

If the real order of Γ is finite, it must be odd. Thus these results are related to work by Juel, Haupt, and others. In fact, theorem 1 has been proved by Juel under weaker assumptions. Theorem 2 does not seem to be contained in the known generalizations of the four-vertex theorem and a more detailed analysis would be desirable. [Cf. P. Montel, *Bull. Sci. Math.* 48 (1924), 109-128; esp. Section 3. Also O. Haupt, *Colloque sur les questions de réalité en géométrie*, Liège, 1955, pp. 59-76, Masson, Paris, 1956; MR 17, 887; esp. p. 65f.] P. Scherk (Saskatoon, Sask.)

6139:

Toponogov, V. A. Riemannian spaces having their curvature bounded below by a positive number. *Dokl. Akad. Nauk SSSR* 120 (1958), 719-721. (Russian)

Soit R_k^m un espace de la classe C^2 avec la métrique complète à courbure $> k$ ($= \text{const} > 0$). Pour la somme S des longueurs des côtés d'un triangle $A_1 A_2 A_3$ géodésique dans R_k^m on a $S \leq 2\pi/\sqrt{k}$; si $S = 2\pi/\sqrt{k}$ et $\overline{A_1 A_2} + \overline{A_1 A_3} \neq \overline{A_2 A_3}$, alors les côtés sont les arcs d'une courbe géodésique fermée et R_k^m est une sphère au rayon $1/\sqrt{k}$. Pour la

longueur l d'un arc géodésique γ on a $l \leq \pi/\sqrt{k}$; l'existence d'un tel γ avec $l = \pi/\sqrt{k}$ signifie que R_k^m est une sphère.

Le théorème "Les angles du triangle géodésique $A_1 A_2 A_3$ dans R_k^m ne sont pas inférieurs aux angles du triangle $A_1' A_2' A_3'$ (avec $\overline{A_1 A_2} = \overline{A_1' A_2'}$) d'un plan R_k à la courbure constante k " a été démontré par l'A. [même Dokl. 115 (1957), 674-676; MR 19, 979] pour le cas $k=0$, ici on le prouve pour $k>0$. A. Švec (Prague)

6140:

Su, Buchin. Koschmieder invariant and the associate differential equation of a minimal hypersurface in a regular Cartan space. *Acta Math. Sinica* 6 (1956), 374-388. (Chinese. English summary)

In a preceding paper [Acta Math. 85 (1951), 99-116; MR 12, 749] the author has considered infinitesimal deformations of the form $\tilde{x}^i = x^i + \epsilon^i(x) \delta t$ as applied to hypersurfaces in a regular Cartan space based on the notion of area. The more particular problem treated by the author was that of expressing the effect upon the hypersurface quantities of the above deviation in terms of the curvature and torsion tensors of the space. In this paper he treats the same problem with respect to the more general deformation obtained by allowing the ϵ^i to be functions not only of the x , but also of the covariant vector density p_i , characterizing the hypersurface element at every point. His conclusions are that the results hold for the more general case. E. T. Davies (Southampton)

6141:

Gu, Čao-hao. Imbedding of Finsler manifolds in a Minkowski space. *Acta Math. Sinica* 8 (1958), 272-275. (Chinese. Russian summary)

The author proves that any n -dimensional manifold of class C^r ($r \geq 2$) may be imbedded in a $4n$ -dimensional affine space such that tangent vectors at different points are not parallel. According to his definition the theorem asserts that any n -dimensional Finsler space may be imbedded in the large in a $4n$ -dimensional Minkowski space. T. K. Pan (Norman, Okla.)

6142:

Freeman, J. G. Complete Finsler-Riemann systems. *Quart. J. Math. Oxford Ser. (2)* 8 (1957), 161-171.

A Finsler-Riemann (F-R) system, consisting of an n -dimensional deformable subspace in a Riemann v -dimensional space ($v=2n-1$) together with trajectories and generators, has first fundamental form given by 'Finsler-Riemann systems' [same J. 7 (1956), 100-109; MR 20# 3581; for terms and notations, cf. this review], viz. $ds^2 = g_{ab}(x, u) dx^a dx^b$ ($a, b=1, 2, \dots, n$), the length L of the elements of support is given by $L^2 = g_{ab} u^a u^b$. The Finsler space having the x^a for coordinates, the u^a for components of support, and the same function L for the length of the elements of support, is called the Finsler image of the F-R system.

Those elements of the Finsler image which may differ from the corresponding elements of the F-R system are distinguished by a vertical bar placed after them (but before the affixes), so that the first fundamental form of the image is denoted by $ds^2 = g|_{ab} dx^a dx^b$, and the absolute differential in the image of a vector with components X^a by $DX|_a = dX^a + x^b dx^c T|_{bc}^a + X^b du^c C|_{bc}^a$. With this notation a normal F-R system (cf. loc. cit.) is one for which $g_{ab} = g|_{ab}$. When $C|_{bc}^a = C|_{bc}^a$ the F-R system is called C-type; when $\Gamma|_{bc}^a = \Gamma|_{bc}^a$ it is called Γ -type; and when $g_{ab}, C|_{bc}^a, \Gamma|_{bc}^a$ equal $g|_{ab}, C|_{bc}^a, \Gamma|_{bc}^a$, respec-

tively, it is called complete when also $\Gamma^*_{bc^a} = \Gamma^*|_{bc^a}$, $A_{bc^a} = A|_{bc^a}$, (where $A_{bc^a} = LC_{bf^a}\phi_c^f$, ϕ_c^f being the minor in $|\delta_c^a + LC_{0c^a}|$) since the formulas [(3.18), (3.19), (3.20)] expressing $\Gamma^*_{bc^a}$, A_{bc^a} in terms of Γ_{bc^a} , C_{bc^a} are then identical with the corresponding formulae for the image. In a complete F-R system the g_{ab} , Γ_{bc^a} , C_{bc^a} , $\Gamma^*_{bc^a}$, A_{bc^a} can be expressed in terms of the derivatives of L^2 only (this being a property of the corresponding expressions in the image). In the last section the existence of a complete F-R system having a given function for the square of the length of its elements of support is considered. S_n -coordinates for which $y^a = x^a$, $y^A = t^A$ are considered. The main results are as follows.

(1) Necessary and sufficient conditions for a system to be normal and of C-type are (i) $C_{0bc} = 0$ and (ii) $C_{abc} = C_{bac}$, where $C_{abc} = g_{bc}C_{ac}^a$. If a system is normal, a necessary and sufficient condition for it to be of C-type is $C_{abc} = C_{bac}$. (2) If the elements of support of a F-R system are transported along trajectories by induced parallelism in S_n with deformation of S_n , then (i) A_{bc^a} and $\Gamma^*_{bc^a}$ are indeterminate, and (ii) the system cannot be of C-type, and therefore cannot be complete. (3) A necessary and sufficient condition for the system to be normal is $g_{ab} = \partial^2 F / \partial u^a \partial u^b$. (4) A sufficient condition that the normal system be of C-type is that $\partial g_{aa} / \partial y^b = \partial g_{ba} / \partial y^a$. (5) Given a function $2F(x, u)$, homogeneous of second order in the u^a , there exist normal, C-type systems in which $L^2 = 2F$. (6) Given a function $2F(x, u)$, homogeneous of second order in the u^a , which either is independent of the x^a or can be made so by a transformation of S_n -coordinates, there exist complete systems in which $L^2 = 2F$. (7) If $n=2$, given $2F(x, u)$, homogeneous of second order in the u^a , there exist complete systems in which $L^2 = 2F$, provided that the equation $r\partial^2/\partial x^1 + \partial^2/\partial x^2 = H(x^1, x^2, r)\partial^2/\partial r$ (which gives the generators) has a solution which is valid for all values of x^1, x^2, r , where $r = u^1/u^2$, $S|_{121}/u^2 C_{111} = H(x^1, x^2, r)$ [for $S|_{abc}$ cf. E. Cartan, *Les espaces de Finsler*, Hermann, Paris, 1934, formula 16].

T. Takasu (Yokohama)

PROBABILITY

See also 5910, 6182, 6325, 6326, 6327.

6143:

*Fortet, Robert. Recent advances in probability theory. Some aspects of analysis and probability, pp. 169-240. Surveys in Applied Mathematics. Vol. 4. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xi+243 pp. \$9.00.

This is a survey article of recent progress in probability theory up to about 1955. It is written in a concise but very lucid style and contains much valuable information on the topics that the author has selected.

After a brief discussion of the role of Kolmogorov's axioms in the foundation of the theory, the author treats the central limit theorem and some of its ramifications: the Berry-Esseen bound of the error, local limit theorems and the case of lattice distributions. The work on stochastic elements with values in an abstract space is reviewed, including the situation when the values are Schwartz distributions. The author discusses applications to von Mises' statistical functions and to stochastic processes. One chapter is devoted to the study of functionals of random functions, especially in the Markov

case, e.g., the results of Kallianpur-Robbins on the asymptotic behavior in the homogeneous case. This is followed by a rather detailed exposition of extensions of the Kolmogorov-Smirnov theorems on empirical distribution functions. In the last chapter a brief survey is made of statistical problems in stochastic processes: extrapolation, estimation and testing hypotheses. A selective bibliography is given on the subjects discussed in the text.

U. Grenander (Stockholm)

6144:

*Fortet, Robert M. Lois des grands nombres pour des éléments aléatoires généraux. Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 360-364. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. \$7.00.

This paper is a concise survey of some of the results obtained (up to 1954) by E. Mourier, S. Doss and the author concerning laws of large numbers for random variables having abstract range spaces. A slightly more recent and detailed survey is given by E. Mourier [Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954/55, vol. 2, pp. 231-242, University of California Press, Berkeley and Los Angeles, 1956; MR 19, 1202].

R. Pyke (New York, N.Y.)

6145:

*Špaček, Antonín. Prolongement des transformations aléatoires. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 259-272. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

Given a pair of metric spaces A and B , the author formulates conditions that a mapping from part of A into B , depending on a stochastic parameter, be extensible, with probability one, to a mapping from the whole of A into B .

H. P. McKean, Jr. (Cambridge, Mass.)

6146:

Fréchet, Maurice. Sur diverses définitions de la moyenne d'un élément aléatoire de nature quelconque. II. Giorn. Ist. Ital. Attuari 20 (1957), 1-37.

Expository paper, continuing an earlier one [same Giorn. 19 (1956), 1-15; MR 19, 325].

J. L. Doob (Urbana, Ill.)

6147:

Studnev, Yu. P. On the role of Lindeberg's conditions. Dopovidi Akad. Nauk Ukrain. RSR 1958, 239-242. (Ukrainian. Russian and English summaries)

Let $\xi_1, \dots, \xi_n, \dots$ be a sequence of independent random quantities with distribution functions $F_1(x), \dots, F_n(x), \dots$. Furthermore, let

$$\sigma_1 = \int_{-\infty}^{\infty} x^2 dF_1(x), \quad B_n^2 = \sum_{i=1}^n \sigma_i^2,$$

$$L_n(x) = B_n^{-2} \sum_{i=1}^n \int_{|x| \geq x} x^2 dF_i(x).$$

In this notation the condition of Lyapunov's theorem, as formulated by Lindeberg and used in this paper, takes the form $\lim_{n \rightarrow \infty} L_n(B_n) = 0$.

The main result of this work can now be stated as the following theorem. Under the condition of Lyapunov's theorem, as formulated by Lindeberg, the following relation holds:

$$\max_{-\infty < x < \infty} |F_n(x) - (2\pi)^{-1} \int_{-\infty}^x e^{-t^2/2} dt| = O(B_n^{-1} \int_0^{B_n} L_n(x) dx) + O(B_n^{-1}).$$

This estimate cannot be improved.

H. P. Thielman (Ames, Iowa)

6148:

Rényi, Alfréd. On the distribution function $L(x)$. Magyar Tud. Akad. Mat. Kutató Int. Közl. 2 (1957), 43-50. (Hungarian. Russian and English summaries)

The author publishes three proofs for the fact that the distribution function

$$L(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2 \pi^2}{8x^2}\right) \quad (x > 0)$$

can be expressed as follows:

$$L(x) = \sum_{k=-\infty}^{\infty} (-1)^k [\Phi((2k+1)x) - \Phi((2k-1)x)].$$

[It is to be remarked that this paper is antedated.]

L. Takács (London)

6149:

Sarkadi, Károly. Generalized hypergeometric distributions. Magyar Tud. Akad. Mat. Kutató Int. Közl. 2 (1957), 59-69. (Hungarian and Russian summaries)

The author shows that the Pólya distribution is a special case of the generalised hypergeometric distribution of C. D. Kemp and A. W. Kemp [J. Roy. Statist. Soc. B. 18 (1956), 202-211; MR 18, 769], and several known distributions (e.g., Laplace's law of succession) are special cases of the Pólya distribution. Finally, it is shown that every random variable with possible values 0, 1, ..., n can be expressed as a sum of n equivalent random variables with possible values 0 and 1.

L. Takács (London)

6150:

Hájek, Jaroslav. A property of J -divergences of marginal probability distributions. Czechoslovak Math. J. 8(83) (1958), 460-463. (Russian summary)

Consider two arbitrary probability distributions P and Q on a Borel field F of subsets Λ of a space $\Omega = \{\omega\}$, and let P_a and Q_a , $a \in A$, be corresponding "marginal" distributions on Borel sub-fields $F_a \subset F$, defined by $P_a(\Lambda) = P(\Lambda)$, $Q_a(\Lambda) = Q(\Lambda)$ for $\Lambda \in F_a$, $a \in A$. J -divergence J_a between distributions P and Q on the Borel field $F_a \subset F$ is the number

$$J_a = \int \left(\frac{p_a}{q_a} - 1 \right) \log \frac{p_a}{q_a} dQ \text{ if } P_a = Q_a,$$

$$J_a = \infty \text{ if } P_a \neq Q_a,$$

where $P_a = Q_a$ denotes that $[Q(\Lambda) = 0] \Leftrightarrow [P(\Lambda) = 0]$ for $\Lambda \in F_a$, and $p_a/q_a = dP_a/dQ_a$ is the Radon-Nikodym derivative of P_a with respect to Q_a . This definition is that of Kullback and Leibler [Ann. Math. Statist. 22 (1951), 79-86; MR 12, 623] extended to the case when $P_a \neq Q_a$ and is a form introduced by Jeffreys [Proc. Roy. Soc. London, Ser. A. 186 (1946), 453-461; MR 8, 163]. The author proves that the J -divergence of any two probability distributions of any stochastic process equals the supremum of J -divergences of finite-dimensional marginal distributions. If this supremum is finite then the distributions are absolutely continuous with respect to each other.

S. Kullback (Washington, D.C.)

6151:

Wintner, Aurel. Indefinitely divisible symmetric laws and normal stratifications. Publ. Inst. Statist. Univ. Paris 6 (1957), 327-336.

A distribution $\phi(x)$ on $-\infty < x < \infty$ is called a strati-

fication of normal symmetric distributions if there exists a unilateral distribution $\mu(x)$ ($\mu(x) = 0$ for $x < 0$) in terms of which $\phi(x) = \int_0^\infty \gamma(x/u) d\mu(u)$. Here $\gamma(x) = (2\pi)^{-1} \int_{-\infty}^\infty \exp(-\frac{1}{2}y^2) dy$ and $\gamma(x/\alpha)$ for $\alpha = 0$ is defined to be the Dirac distribution $\frac{1}{2} + \frac{1}{2} \operatorname{sgn} x$. It is shown, among other things, that a symmetric, indefinitely divisible distribution ϕ is a normal stratification if and only if there exists a unilateral distribution $\lambda(x)$ satisfying $\int_1^\infty x^{-1} d\lambda(x) < \infty$ and such that its Fourier transform $F(t) = \int_{-\infty}^\infty e^{itx} d\lambda(x)$ has the form $F(t) = \exp[-\int_0^\infty x^{-1} (1 - e^{-ix}) d\lambda(x)]$, where both cases $\lambda(+0) \geq \lambda(0)$ are allowed. A characterization of a class of distributions considered by Lévy [Bull. Sci. Math. 63 (1939), 247-268; MR 1, 149] is obtained. Also considered are stable distributions (with Fourier transform $F(t) = \exp(-|t|^p)$, $p \geq 0$) and stratifications of these, a generalization of the symmetric normal stratifications.

C. R. Putnam (Lafayette, Ind.)

6152:

Lomnicki, Z. A.; and Zaremba, S. K. A further instance of the central limit theorem for dependent random variables. Math. Z. 66 (1957), 490-494.

Consider a set of stochastic variables ξ_{ik} ; $i = 1, 2, \dots$; $k = i+1, i+2, \dots$; with mean zero and constant variance. If any two finite sets of ξ_{ik} have the property that no value taken by either index of any element of one set appears among the values of the indices of the elements of the other set, then these two sets are mutually independent. The authors assume further that $E\xi_{ik}\xi_{ij} = E\xi_{ij}\xi_{ik} = \text{constant } c$ for any $i < k < j$ and that the variance integrals converge uniformly in i and k . Then the sum

$$\frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{k=i+1}^N \xi_{ik}$$

is asymptotically normal with mean zero and variance $4c$. An application is made to Gini's mean difference.

U. Grenander (Stockholm)

6153:

Franckx, Ed. La loi forte des grands nombres des variables uniformément bornées. Critère des sous-suites caractéristiques. Trabajos Estadist. 9 (1958), 111-115. (Spanish summary)

Let $\{X_n\}$ be uniformly bounded random variables with $S_n = \sum_{i=1}^n X_i$. Then $P(\lim_{n \rightarrow \infty} n^{-1}(S_n - ES_n) = 0) = 1$ if and only if $P(\lim_{n \rightarrow \infty} n_i^{-1}(S_{n_i} - ES_{n_i}) = 0) = 1$ for some subsequence of the integers with $\lim_{n \rightarrow \infty} n_i^{-1} \cdot n_{i+1} = 1$.

H. Teicher (Lafayette, Ind.)

6154:

Cartwright, D. E. The prediction of a random function, given simultaneous values of its first few derivatives. J. Math. Phys. 37 (1958), 229-245.

The author considers a stationary Gaussian process, $x(t)$, and calculates the conditional distribution, mean and variance of $x(t+\tau)$, given x and some of its derivatives at time t .

The conditional distribution is not Gaussian, on account of the interesting feature, not fully explained, that the conditioning relation is: $x = x_0$ somewhere in $(t, t+\delta t)$, rather than: $x_0 \leq x(t) \leq x_0 + \delta x$.

P. Whittle (Wellington)

6155:

Takács, L. On a probability problem in theory of counters. Ann. Math. Statist. 29 (1958), 1257-1263.

The model proposed by G. E. Albert and Lewis Nelson [Ann. Math. Statist. 24 (1953), 9-22; MR 14, 775] for reconciliation of Type I and Type II representation of Geiger-Mueller and scintillation counters, described roughly by the prescription that particles arriving during a counter dead time are not counted but extend the dead

time with probability p , is extended to allow an arbitrary distribution function for the (unextended) dead time; the particle arrivals, as with Albert and Nelson, are supposed to form a Poisson process with constant density. The essential variable is the distribution function for the interval between successive registrations; its Laplace-Stieltjes transform is given explicitly as are its mean and variance. The relation of the variable of chief physical interest, the number counted in a given time interval, to this variable is given by a number of general theorems. It is remarked that the further extension to a general interarrival time distribution has already appeared in previous papers of the author.

J. Riordan (New York, N.Y.)

6156:

Snell, J. Laurie. Finite Markov chains and their applications. Amer. Math. Monthly 66 (1959), 99-104.

Basic facts about finite Markov chains. The (Green) matrix $(n_{ij}) = (\text{expected number of visits to } j \text{ starting at } i)$, $i, j \leq n$, is made to play a central role. A monograph on the subject (in collaboration with J. G. Kemeny) is promised for the near future.

H. P. McKean, Jr. (Cambridge, Mass.)

6157:

Kinney, John R. Singular functions associated with Markov chains. Proc. Amer. Math. Soc. 9 (1958), 603-608.

Let $\{x_i\}$ be a Markov chain with states $0, 1, \dots, M-1$ and the stationary transition matrix $\|a_{ij}\|$, a single ergodic class, and stationary probabilities b_i . Let $\alpha = -\sum_{i,j} b_i a_{ij} \lg_M a_{ij}$, a number proportional to the entropy. Let $F(x) = \Pr\{\sum_{i=0}^{\infty} x_i M^{-i} < x\}$. It is shown that F is a singular distribution whose mass is concentrated on a null set of Hausdorff dimension α . A result by H. G. Eggleston [Proc. London Math. Soc. (2) 54 (1953), 42-93; MR 14, 23] is deduced from this. There is another result identifying the capacity of a binary channel with the Hausdorff dimension number with the "message set".

K. L. Chung (Syracuse, N.Y.)

6158:

Bellman, Richard; and Kalaba, Robert. Random walk, scattering, and invariant imbedding. I. One-dimensional discrete case. Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 930-933.

The inhomogeneous version of the gambler's ruin problem is solved by a functional equation.

J. Wolfowitz (Ithaca, N.Y.)

6159:

Malécot, G. Sur quelques processus de "mouvement brownien". Ann. Univ. Lyon. Sect. A (3) 20 (1957), 33-53.

The author discusses the following "Brownian motion". A massive particle moves in one dimension subject to a restoring force proportional to the displacement from a central point on its path, and its motion is perturbed by a sequence of random shocks acting at random epochs. If $X(t)$ and $Y(t)$ denote the velocity and displacement (respectively) at the epoch t , then the author imposes the condition that there is a chance $\mu_i \Delta t + o(\Delta t)$ of a velocity-increment due to a shock of amount $\alpha_i = \phi_i(v_i - X(t))$ during the interval $(t, t + \Delta t)$ (such an event corresponding to a collision between the Brownian particle whose motion is being studied and a perturbing particle of velocity v_i). (Here i ranges through a denumerable set, but the author envisages also the natural continuous analogue.) With these hypotheses the author studies the means, variances and covariance of the speed

and location of the Brownian particle at a general epoch. A concluding note deals with the case of a linear repulsion from a fixed centre (instead of the restoring force).

D. G. Kendall (Oxford)

6160:

Prabhu, N. U. Some exact results for the finite dam. Ann. Math. Statist. 29 (1958), 1234-1243.

The author studies the integer valued Markov chain $\{Z_n: n \geq 1\}$ which represents the storage of a dam of finite capacity K where just prior to the n th unit of time, a unit of water is released and just after the n th unit of time an amount X_n is put into the dam. In this paper $\{X_n: n \geq 1\}$ are independent and identically distributed positive-integer-valued random variables. It is seen that $Z_n = \max\{Z_{n-1} + X_n - 1, 0\}$. The stationary probabilities are obtained for this chain when X_n has either a (i) geometric, (ii) negative binomial, or a (iii) Poisson distribution. A discussion is also given of the probability of emptiness before an overflow occurs. Generating function methods are employed.

R. Pyke (New York, N.Y.)

6161:

Tanner, J. C. A simplified model for delays in overtaking on a two-lane road. J. Roy. Statist. Soc. Ser. B. 20 (1958), 408-414.

The author imagines a pattern of traffic flow on a two lane highway in which all cars in one lane travel with the same speed v , all cars in the second lane travel in the opposite direction with a speed V and in each lane the distance between any two cars is random with an exponential distribution. He then adds to the first lane a single car with a free speed $u > v$ and assumes that whenever this car wishes to pass another car in its lane, it can do so immediately with no delay if there is a gap of length d in the opposing lane. Otherwise, it must travel at the speed v until it finds a gap of length D , $D > d$, in the opposing lane and passes after an acceleration period during which the gap diminishes by an amount $D - d$.

One interesting feature of the problem is that when the fast car passes another car, it is known that the gap in the opposing lane is at least d , if the fast car then overtakes still another car before it has traveled a distance d relative to the opposing traffic, one still has some knowledge about the size of the gap in the opposing lane. The effect of this is included in the calculations. Its size, however, is probably comparable with the effects of non-zero lengths of the cars which is not considered.

G. Newell (Stockholm)

6162:

Nelson, Ross T. Waiting-time distributions for application to a series of service centers. Operations Res. 6 (1958), 856-862.

The author considers the steady-state behavior of a network of waiting lines with Poisson sources and exponential waiting times as a model for a multistep production process. His results are based on the known fact [Burke, Jackson, Reich] that the steady-state output of a queue with Poisson input and exponential time is again Poisson-like.

E. Reich (Minneapolis, Minn.)

6163:

Kac, M. Some remarks on stable processes. Publ. Inst. Statist. Univ. Paris 6 (1957), 303-306.

The author considers the process with stationary independent increments $\{x(\tau): \tau \geq 0\}$ (with $x(0) = 0$) such that

$$E\{e^{i\lambda x(\tau)}\} = e^{-\tau|\lambda|^2}$$

where λ is real and $0 < \alpha \leq 2$, and studies for $a > 0$ the probability $P_a(t; a)$ that either $x(\tau-0) = a$ or $x(\tau+0) = a$ for some epoch τ in the interval $0 < \tau \leq t$. He remarks that $P_a(t; a)$ is known for $0 < \alpha \leq 1$ and for $\alpha = 2$, and consequently this paper is concerned only with the case $1 < \alpha < 2$. The main result is that

$$\int_0^\infty e^{-\lambda t} P_a(t; a) dt = K_a(a; \alpha) / K_a(0; \alpha),$$

where

$$K_a(x; \alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos xu}{s + |u|^\alpha} du.$$

He solves a similar problem concerning the probability $R_a(t; a; b)$ that $x(\tau \pm 0) \neq a$ and $x(\tau \pm 0) \neq b$ for all epochs τ in the interval $0 < \tau \leq t$ (when $0 < a < b$) and obtains an asymptotic estimate of the form $R_a(t; a; b) \sim C_a(a; b) t^{-1/\alpha}$ ($t \rightarrow \infty$) by a tauberian argument.

D. G. Kendall (Oxford)

6164:

Prabhu, N. U. On the integral equation for the finite dam. Quart. J. Math. Oxford Ser. (2) 9 (1958), 183-188.

X_t ($t=0, 1, 2, \dots$) represents the inputs which flow into a dam of capacity k in the respective intervals $(t, t+1)$; X_t are independently and identically distributed, with varying t ; Z_t ($t \leq k$) represents the storage at time t . U_{t+1} is the amount of water released at time $t+1$:

$$U_{t+1} = \begin{cases} m & \text{if } X_t + Z_t > m, \\ X_t + Z_t & \text{if } X_t + Z_t \leq m, \end{cases}$$

where m ($< k$), as well as k , is given.

Let $G(x)$ be the stationary distribution function of the input X_t , and $H(y)$ the unknown stationary distribution function of $Y_t = Z_t + X_t$. $H(y)$ satisfies the integral equation of Moran (whose papers were the starting point of the present work):

$$H(y) = \begin{cases} -\int_m^{m+y} H(t) dG(m+y-t) & (y < k-m) \\ G(y-k+m) - \int_m^k H(t) dG(m+y-t) & (y \geq k-m). \end{cases}$$

The author obtains an exact solution $H(y)$ of these equations, provided that $G(x)$ is of the general gamma type

$$dG(x) = \frac{\mu^p}{(p-1)!} e^{-\mu x} x^{p-1} dx \quad (0 < x < \infty).$$

The release rule is also obtained, in quite good agreement with the expectances.

O. Onicescu (Bucarest)

6165:

Scheffler, H. Strahlenoptische Ausbreitung in Medien mit statistisch verteilten Inhomogenitäten. I. Unregelmässige Refraktion als Markoff-Prozess. Astr. Nachr. 284 (1958), 227-232.

Making the approximation of ray optics, the author studies the propagation of electromagnetic radiation in a medium with random inhomogeneities of refractive index. The ray deflections and ray directions are regarded as Markov processes with the distance penetrated into the medium as parameter. The Fokker-Planck equation is set up for the distribution of ray directions, and then solved for the special case of propagation in a plane. A similar approach to the problem of ray propagation in a randomly inhomogeneous medium can be found in the work of V. Ya. Haranen [Dokl. Akad. Nauk SSSR 88 (1953), 253-256; MR 14, 923] and in L. A. Černov's book: "Raspro-

stranenie voln b srede so slučajnymi neodnorodnostyami" [Izdat. Akad. Nauk SSSR, Moscow, 1958].

R. A. Silverman (New York, N.Y.)

STATISTICS

See also 6344.

6166:

Gumbel, Émile J. Distributions à plusieurs variables dont les marges sont données. C. R. Acad. Sci. Paris 246 (1958), 2717-2719.

6167:

Fréchet, Maurice. Remarques au sujet de la note précédente. C. R. Acad. Sci. Paris 246 (1958), 2719-2720.

The author comments on the previous note [#6166 above] and announces three forthcoming publications applying his results [Ann. Univ. Lyon. Sect. A. (3) 14 (1951), 53-77; MR 14, 189], which are of the type: for distribution functions H, F_i ,

$$1 - \sum [1 - F_i(x_i)] \leq H(x_1, \dots, x_n) \leq \min F_i(x_i)$$

for all x_1, \dots, x_n if and only if the F_i are the marginals of H .

J. Hannan (East Lansing, Mich.)

6168:

Fréchet, Maurice. Les tableaux de corrélation dont les marges et des bornes sont données. Ann. Univ. Lyon. Sect. A (3) 20 (1957), 13-31. (Esperanto summary)

The author studies the class of 2-rowed matrices of non-negative integers with given compatible marginal totals N_i, N_j' and given bounds m_{ij} . When the essential bounds, $\min(m_{ij}, N_i, N_j')$, are compatible with the N_i and N_j' , the then non-empty class is constructed and further examined.

J. Hannan (East Lansing, Mich.)

6169:

Maniya, G. M. Quadratic error of estimation of density of a normal two-dimensional distribution in terms of sampling data. Soobšč. Akad. Nauk Gruzin. SSR 20 (1958), 655-658. (Russian)

6170:

St-Pierre, Jacques. Distribution of linear contrasts of order statistics. Ann. Math. Statist. 29 (1958), 1264-1268.

Let X_0, X_1, \dots, X_n be $n+1$ independent normal random variables with constant variance and expectations $\mu_0, \mu_1, \mu_2, \dots, \mu_n$ respectively. If $X_{(0)} \geq X_{(1)} \geq \dots \geq X_{(n)}$ are the corresponding order statistics, then a linear contrast of these order statistics is defined by

$$Z = X_{(0)} - \sum_{i=1}^n c_i X_{(i)}, \quad \sum_{i=1}^n c_i = 1, \quad 0 \leq c_i \leq 1 \quad (i=1, \dots, n).$$

Formulas and/or tables for the probability density of Z are obtained in the case of $n=2$ under the hypotheses: $H_0: \mu_0 = \mu_1 = \mu_2 = 0$; $H_1: \mu_0 = \delta, \mu_1 = \mu_2 = 0$ ($\delta > 0$); and $H_2: \mu_0 = 2, \mu_1 = 1, \mu_2 = 0$. F. C. Andrews (Eugene, Ore.)

6171:

Okamoto, Masashi. Some inequalities relating to the partial sum of binomial probabilities. Ann. Inst. Statist. Math. 10 (1958), 29-35.

In Uspensky, "Introduction to Mathematical Probab-

ility" [McGraw-Hill, New York, 1937], the following inequality appears involving the random variable X with the binomial distribution $B(n, p)$: $P\{|X/n - p| \geq c\} < 2 \exp(-nc^2/2)$. The author proves several inequalities, the simplest of which is $P\{|X/n - p| \geq c\} < \exp(-2nc^2)$, which is an improvement. These inequalities are applied to find a new bound for the distribution of Matusita's multinomial distance which can be used as a goodness of fit statistic. (The reviewer believes that the main part of page 32 may be omitted by noting that the proof of (d) in case (i) implies equation (12). In the statements of theorems 1 to 4, π should be replaced by X .)

H. Chernoff (Stanford, Calif.)

6172:

Welch, B. L. "Student" and small sample theory. J. Amer. Statist. Assoc. 53 (1958), 777-788.

This article reviews W. S. Gosset's work on the occasion of the fiftieth anniversary of the publication of "Student's" distribution.

6173:

Garner, Norman R. Curtailed sampling for variables. J. Amer. Statist. Assoc. 53 (1958), 862-867.

A lot is tested for acceptance under a variables sampling acceptance plan with a single sample of n and the acceptance criterion that $\bar{X} + Ks \leq L$ in which \bar{X} is the sample mean, s^2 is the unbiased estimate of the lot variance, and K is a constant determined by the specifications. It may occur that after measurements, X , have been obtained in the first r items in the sample, the mean, \bar{X}_r , and variance, V_r , of the sub-sample of r are such that the lot will be rejected no matter what the results are for the remaining $n-r$ items. By determining the set of values of the $n-r$ X 's most favorable for acceptance, the author determines in terms of n , r , L , \bar{X}_r , and V_r a criterion for curtailing sampling inspections at r items, the lot at that point being certain to be rejected.

C. C. Craig (Ann Arbor, Mich.)

6174:

Cohen, Leonard. On mixed single sample experiments. Ann. Math. Statist. 29 (1958), 947-971.

Let X_1, \dots, X_n be identically, independently distributed with distribution P_0 or P_1 and let an admissible test of P_0 against P_1 have error probabilities α and β . It was pointed out by Kruskal (in an unpublished lecture in 1954) that a random choice of sample size according to known probabilities may lead to a smaller expected sample size without increasing either of the (expected) probabilities of error. In the present paper, a complete class of such randomized procedures is characterized. The result is applied to the problems of testing the mean of a normal distribution with known variance, of a binomial distribution, of a rectangular distribution over $(0, \theta)$ and of a rectangular distribution over $(\theta, \theta+1)$. In all except the last of these cases improvement is possible through randomization at least for certain values of α , β and n . The author discusses more briefly also the corresponding question for certain other classes of problems including estimation by confidence intervals and k -decision problems.

E. Lehmann (Berkeley, Calif.)

6175:

Tucker, Ledyard R. An inter-battery method of factor analysis. Psychometrika 23 (1958), 111-136.

How well can factors be identified between several analyses when different test batteries are given to one sample of individuals? Stressing that this important aspect of factor reliability has not been touched by

hypothesis testing, the author makes an interesting approach based on the criterion that the scores for individuals on a factor should remain invariant as the individuals are tested with different batteries which involve the factor. The author develops a technique for the case of two test batteries, postulated to depend on the same common factors. The factors, say with loadings matrix A_1 for battery 1 and A_2 for battery 2, are estimated from the relation $R_{12} = A_1 A_2^*$, where R_{12} is the matrix of correlations of the scores in one battery with the scores in the other. No communalities estimates are required. Using Eckhart-Young's method [Psychometrika 1 (1936), 211-218] for least squares approximation of one matrix by another of lower rank u , the loadings A_1, A_2 are estimated by fitting $A_1 A_2^*$ to R_{12} for any desired u (rank = number of common factors). A heuristic statistical test is provided for the minimum of u . A final step, which involves a generalization of Young-Whittle's factor technique [P. Whittle, Skand. Aktuarietidskr. 35 (1952), 223-239; MR 17, 872], provides pairs of loadings vectors and of corresponding factors, with rotation of axes carried out independently in the two batteries, and the factors for each battery coming out as composite test scores. Pairwise matching composites in the two batteries have maximum covariance, but otherwise all factors are orthogonal. The correlations of matching factors are taken as factor reliability coefficients. An empirical illustration involving 4 tests in the first battery and 5 tests in the second gives $u=2$ factors with reliability coefficients .693 and .821.

H. Wold (New York, N.Y.)

6176:

van Eeden, Constance; and Benard, A. A general class of distributionfree tests for symmetry containing the tests of Wilcoxon and Fisher. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. A. 60=Indag. Math. 19 (1957), 381-408.

Consider m independent random variables z_1, \dots, z_m . A class of tests of the hypothesis that the distribution functions of z_1, \dots, z_m are symmetric about 0 is defined as follows. Let $0 < u_1 < \dots < u_k$ be the absolute values of the non-zero observations on z_1, \dots, z_m . Let a_i and b_i denote the number of observations which are equal to $+u_i$ and $-u_i$, respectively. The test statistic is then given by $T = \sum_{i=1}^k \phi_i (a_i - b_i)$, where the ϕ_i are weights depending on the u_i and/or their ranks. T is a linear combination of the generalized Wilcoxon 2-sample statistic and the sign-test statistic. The distribution of T is investigated as well as conditions for the consistency of the test. Finally, the test based on T is combined with the sign test.

G. E. Noether (Boston, Mass.)

6177:

Karlin, Samuel. Pólya-type distributions. III. Admissibility for multi-action problems. Ann. Math. Statist. 28 (1957), 839-860.

[For part II, see same Ann. 28 (1957), 281-308; MR 19, 475.] The author is concerned with the admissibility of decision procedures in the monotone likelihood ratio case. If the loss function satisfies a certain monotonicity requirement, it was shown by Karlin and the reviewer [same Ann. 27 (1956), 272-299; MR 18, 425] that the class of monotone procedures is complete, and also that not all such procedures are admissible. For all $\omega \in S_j$, the region where the j th procedure is best, let $b_j = |L_j(\omega) - L_{j+1}(\omega)|$ be independent of ω , and

$$\left| \frac{b_j}{b_{j+1,j}} - \frac{b_{j+1,k}}{b_{j+1,k}} \right| \geq 0$$

if $1 \leq j \leq i < i+1 \leq k \leq n$. Then all non-degenerate monotone procedures are admissible; an important special case is $L_1(\omega) = a|j-i|$ for $\omega \in S_j$.

Some results are obtained concerning degenerate procedures in general. If the above determinant is non-negative whenever $1 \leq j < k \leq n$, and b_{ij} is always non-zero, all monotone procedures are admissible. Also, if the underlying measure is atomless, all monotone procedures are admissible. By a limiting argument, it is shown that in the estimation problem, with $L(a, \omega) = c|a - \omega|$, all monotone bounded continuous procedures are admissible.

Another case in which admissibility for all monotone procedures is demonstrated is that in which $L_1(\omega)$ is a monotone decreasing function for $\omega \in S_j$, $i < j$, the L 's are distinct under this condition, and $L_1(\omega) = c > 0$ if $\omega \in S_j$, $j < i$.
H. Rubin (Eugene, Ore.)

6178:

Karlin, Samuel. Pólya type distributions. IV. Some principles of selecting a single procedure from a complete class. *Ann. Math. Statist.* 29 (1958), 1-21.

The terminology is as in the previous paper unless otherwise stated. The author considers the following principle of selecting an admissible decision procedure: (M.P.) If $\omega \in S_j$ and $\omega' \in S_j$ then $P(i|\omega) \geq P(i|\omega')$, where $P(i|\theta)$ is the probability of making decision i if θ is the state of nature. In the case of the two-action problem, any monotone procedure is unbiased and hence M.P., and for the 3-, 4-, and 5-action problems, if the problem is strictly of Pólya type 3 [Proc. Third Berkeley Symposium on Probability and Statist., vol. 1, Univ. of California Press, 1956, pp. 115-128; MR 18, 947] the problem is reduced to a one-parameter family. He also considers unbiasedness in the sense of Lehmann, $E[L(\omega, \phi)] \geq E[L(\theta, \phi)]$ for all ω and θ [same Ann. 22 (1951), pp. 587-592; MR 13, 854], and shows that if

$$\left| \begin{array}{cc} b_{ij} & b_{ik} \\ b_{i+1,j} & b_{i+1,k} \end{array} \right| \geq 0$$

for $1 \leq j \leq i < i+1 \leq k \leq n$, there is a unique unbiased procedure, and this procedure is non-degenerate.

H. Rubin (Eugene, Ore.)

6179:

Nisio, Makiko. Note on the truncated sequential games for stochastic processes. *Math. Japon.* 4 (1956), 49-54.

The author generalizes some results of Dvoretzky, Kiefer and Wolfowitz [Ann. Math. Statist. 24 (1953), 254-264, 403-415; MR 14, 997; 15, 242] on sequential decision procedures for processes with continuous time parameter. To this end he studies the Bayes risk of a truncated sequential game. It is assumed that the parameter and action spaces are compact, and that the loss function is continuous (when the action space is finite, the continuity is replaced by boundedness).
A. Dvoretzky (Jerusalem)

6180:

Hannan, E. J. The asymptotic powers of certain tests of goodness of fit for time series. *J. Roy. Statist. Soc. Ser. B* 20 (1958), 143-151.

In an introductory section the author uses the circular model to motivate the spectral theory of general (non-circular) stationary processes. The periodogram and its asymptotic properties are discussed. It is shown that some tests suggested in the literature for testing goodness of fit (autoregressive or moving average hypotheses) can be phrased in a convenient way in terms of the spectral density and the periodogram. The author studies how the

asymptotic power of these tests can be expressed in certain metrics, whose definitions of distance are quadratic functionals of the spectrum. A comparison is made between the various tests.
U. Grenander (Stockholm)

6181:

Broadbent, S. R. The inspection of a Markov process. *J. Roy. Statist. Soc. Ser. B* 20 (1958), 111-119.

Consider items produced in a sequence where 1's indicate good items and 0's defective items. The sequence is assumed to be a simple Markov process in which the probability that an item is bad depends only on the previous item. The transition matrix is $\begin{pmatrix} 1-X & Y \\ X & 1-Y \end{pmatrix}$, where the probability that any item is a 1 depends only on the previous item, and is $1-X$ if preceded by 1, and Y if preceded by 0. The author finds the maximum likelihood estimates of $P=X/(X+Y)$ and $A=1-(X+Y)$. His discussion of the measurement of the errors of inspectors in sorting good or bad items depends on Shannon's theory of information.

L. A. Aroian (Culver City, Calif.)

6182:

Goodman, Leo A. Asymptotic distributions of "psi-squared" goodness of fit criteria for m -th order Markov chains. *Ann. Math. Statist.* 29 (1958), 1123-1133.

Consider a sequence of N symbols each of which can take one of the values $1, 2, \dots, s$. Let f_u be the frequency of the m -tuple (u_1, u_2, \dots, u_m) in the sequence. Let H_n' be the composite hypothesis that the sequence arose from a Markov chain of order n . Let H_n be any simple hypothesis belonging to H_n' . Let H_n^* be the maximum-likelihood H_n . Let the expectation of f_u in a new sequence of length N , given H_n , be $f_{u,n}$, and, given H_n^* , be f_{u,n^*} . Let

$$\psi_{m,n}^2 = \sum_u (f_u - f_{u,n})^2 / f_{u,n}, \quad \psi_{m,n^*}^2 = \sum_u (f_u - f_{u,n^*})^2 / f_{u,n^*},$$

$$\psi_{n+1,n}^2 = 0.$$

The reviewer, when previously reviewing a paper by P. Billingsley [same Ann. 27 (1956), 1123-1129; MR 18, 607], conjectured that the asymptotic distribution ($N \rightarrow \infty$) of $\psi_{m,n}^2$, when H_n' is true, is

$$\sum_{i=1}^{m-n-1} K_{\theta} \omega(x/\lambda),$$

where $*$ denotes convolution, $g(\lambda) = (s-1)^2 s^{m-1-\lambda}$, and $K_i(x)$ is the χ^2 -distribution with i degrees of freedom; and also conjectured the distribution of $\psi_{m,n}^2$ when H_n is true. The author, among other things, proves the first conjecture. He disproves the second one, but proves a modified version of it. He states that closely related results have been proved by Billingsley in an unpublished paper, using entirely different methods.

I. J. Good (Teddington)

6183:

Davies, Hilda M.; and Jowett, G. H. The fitting of Markoff serial variation curves. *J. Roy. Statist. Soc. Ser. B* 20 (1958), 120-142.

Jowett's serial variation statistic $d(s) = \sum_{i=1}^{T-s} \frac{1}{2} (x_i - x_{i+s})^2 / (T-s)$ [cf. *Biometrika* 42 (1955), 160-169; MR 16, 1134] is exploited to graduate the autocorrelation coefficients r_k of not too high order k by $r_k \approx a(1 - b^k)$. The emphasis is on estimating a and b from observed values $d(s)$ with $s=1, \dots, 5$ or $s=1, \dots, 5, 10, 15$, using a routine with auxiliary tables. There are also an approximate test for goodness of fit and empirical illustrations.
H. Wold (New York, N.Y.)

NUMERICAL METHODS

See also 5938, 5978, 6196, 6212, 6227, 6229, 6318.

6184:

Vernotte, Pierre. A propos des notations différentielles en physique mathématique. Actes des colloques de calcul numérique, Caen, 1955; Dijon, 1956, pp. 111-119. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 77, Paris, 1958. vi+144 pp. 2105 francs.

The author points out that it is absolutely necessary, especially for the students, to use different notations for partial and for total differentiation. This is illustrated by examples, taken from differential equations of mathematical physics. *M. J. O. Strutt (Zürich)*

6185:

Ostrowski, Alexander M. On Gauss' speeding up device in the theory of single step iteration. Math. Tables Aids Comput. 12 (1958), 116-132.

Let the system (1) $\sum_{\mu=1}^n a_{\mu\nu} x_{\mu} = y_{\nu}$ ($\nu=1, \dots, n$) be given with the matrix $A=(a_{\mu\nu})$ real, symmetric and positive definite. Consider the extended (singular) system (2) $\sum_{\mu=0}^n a_{\mu\nu} x_{\mu} = y_{\nu}$ ($\nu=0, 1, \dots, n$), of which the last n equations are obtained by introducing $x_{\nu} = z_{\nu} - z_0$ into (1) and the first equation is obtained by taking the negative sum of the others. Gauss' device [Werke, Bd. 9, Teubner, Leipzig, 1903, 278-281; for an annotated translation by G. E. Forsythe see Math. Tables Aids Comput. 5 (1951), 255-258; see also R. Dedekind, "Gesammelte mathematische Werke", Bd. 2, Vieweg, Braunschweig, 1931, 293-306, especially 300-301] consists in applying the single step iteration to (2) rather than (1). If $\xi_k = (z_0^{(k)}, z_1^{(k)}, \dots, z_n^{(k)})$ is the resulting sequence of vectors and $\xi_k = (z_1^{(k)} - z_0^{(k)}, \dots, z_n^{(k)} - z_0^{(k)})$, then the question arises whether the sequence ξ_k converges faster than the sequence which would result if the iteration were applied directly to (1). Opinions in the literature are contradictory [R. Zurmühl, "Matrizen", Springer, Berlin, 1950; MR 12, 73; 280 ff; G. E. Forsythe and T. S. Motzkin, Math. Tables Aids Comput. 6 (1952), 9-17; MR 13, 91].

The author first analyzes the cyclic single step iteration. He derives elegantly the characteristic equation for Gauss' modified procedure, the dominant root of which determines the rate of convergence. For $n=2$ this rate is compared with that of the ordinary cyclic iteration. It is shown, in particular, that there are cases in which the modified procedure is faster and cases in which it is slower than the original procedure. The same situation is proved to be true for certain systems of any order $n > 2$. In the second half of the paper the author discusses from a probabilistic point of view a relaxation single step iteration (Seidel's relaxation). He proves that under certain conditions on the matrix A and the sequence of residual vectors there is a positive probability that Gauss' device would speed up the relaxation procedure.

Walter Gautschi (Washington, D.C.)

6186:

Bolie, Victor W. Minimum-storage matrix inversion. Z. Angew. Math. Mech. 38 (1958), 369-372. (German, French and Russian summaries)

A routine is described which, for inverting a matrix of order n , requires n cells for transient storage, and ultimately leaves the inverse stored in place of the original matrix. By a cyclic interchange of rows and columns the

pivotal element is, in effect, always to be found in the upper left-hand corner. The code does not provide for a search for a pivotal element.

A. S. Householder (Oak Ridge, Tenn.)

6187:

Bondarenko, P. S. Convergence of an algorithm of successive approximations and error estimates in numerical solution of infinite systems of linear algebraic equations. Kifv. Derž. Univ. Nauk Zap. 16 (1957), no. 2=Kiev. Gos. Univ. Mat. Sb. 9 (1957), 81-89. (Russian)

With the system

$$(I) \quad x_i = \sum_{k=1}^{\infty} a_{ik} x_k + b_i \quad (i=1, 2, \dots)$$

is associated a "dominant system"

$$(II) \quad X_i = \sum_{k=1}^{\infty} A_{ik} X_k + B_i,$$

where $|a_{ik}| < A_{ik}$, $|b_i| < B_i$. Starting with a certain set of initial values for the unknowns, $x_i^{(0)}$, one has the iteration $x_i^{(n)} = \sum_{k=1}^{\infty} a_{ik} x_k^{(n-1)} + b_i$. It is shown that this iteration is convergent with a solution of (I) as limit if the dominant system (II) has a non-negative solution (i.e. $X_i \geq 0$) and if the initial values satisfy the condition $|x_i^{(0)}| < KB_i$ with a positive constant K . Another theorem refers to a "regular" system (I) with non-negative coefficients a_{ik} , where $|b_i| < K(1 - \sum_{k=1}^{\infty} a_{ik})$. It is remarked that this theorem disproves a statement in a paper by Koyalovič [Izv. Mat. Inst. W. A. Steklov, 3 (1930), 41-67; in part, p. 51]. The last section of the paper discusses a method of Kantorovič [cf. Kantorovič and Krylov, "Approximative methods of higher analysis", 3rd ed., Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad 1950; MR 13, 77; p. 44-48]. *H. Schwerdtfeger (Montreal, P. Q.)*

6188:

Clement, Paul A. A class of triple-diagonal matrices for test purposes. SIAM Rev. 1 (1959), 50-52.

6189:

Tlegenov, K. B. On solvability of a polynomial by a homogeneous operational cycle. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. no. 6(10) (1957), 97-107. (Russian. Kazah summary)

Let $f(x)$ be a polynomial of n th degree for which, at x_0 , the k th order differences (with difference h) are known ($k=1, 2, \dots, n$). Let U be the matrix with zeros below the main diagonal and ones everywhere else. The author gives a column vector Y_0 involving the differences, such that the first component of the vector $Y_k = U^k Y_0$ is $f(x_0 + kh)$ for $k=1, 2, 3, \dots$.

A. W. Goodman (Lexington, Ky.)

6190:

Hoissommer, D. J. Note on the computation of the zeros of functions satisfying a second order differential equation. Math. Tables Aids Comput. 12 (1958), 58-60.

P. Wynn [see Math. Tables Aids Comput. 10 (1956), 97-100; MR 18, 418] pointed out that if $f(x)$ satisfies a second order differential equation, this fact may be used in computing the zeros of $f(x)$. For $f'' = 2P(x)f' + Q(x)f + 2S(x)$ Wynn's formula leads to the iteration $x_{k+1} = x_k - D^{-1}$, with $D = f'(x_k)/f(x_k) - P(x_k) - S(x_k)/f'(x_k) - \frac{1}{2}Q(x_k)/f(x_k)/f'(x_k)$. The author shows that in D the last term may be omitted without affecting the (cubical) order of the iteration process. *A. J. Kempner (Boulder, Colo.)*

6191:

Hammer, Preston C. The midpoint method of numerical integration. *Math. Mag.* 31 (1957/58), 193-195.

A comparison is given between the midpoint and trapezoidal methods of numerical integration. It is shown that the midpoint method has less error if the second derivative does not change sign in the interval of integration. Simpson's rule is shown to result from a weighted average of the midpoint and trapezoidal rules, giving weight $2/3$ to the former and $1/3$ to the latter.

S. Levy (Philadelphia, Pa.)

6192:

Velasco de Pando, Manuel. The method of regula falsi for solving integral equations. *Dyna* 1956, no. 4, 3-4; no. 9, 2-3. (Spanish)

6193:

Pentkovskii, M. V. Small projective transformations of nomograms. *Dokl. Akad. Nauk SSSR* 121 (1958), 805-806. (Russian)

The author considers nomograms for equations with three variables in which two of the three scales are parallel lines of the same length. The purpose of the paper is to show how a nomogram can be improved by changing the scales without a change of the length of the parallel lines.

S. Kulik (Logan, Utah)

6194:

Salzer, Herbert E.; and Levine, Norman. Table of integers not exceeding 10 00000 that are not expressible as the sum of four tetrahedral numbers. *Math. Tables Aids Comput.* 12 (1958), 141-144.

A tetrahedral number has the form $n(n+1)(n+2)/6$, for positive integral n . The table gives all integers less than one million which cannot be expressed as the sum of four such numbers. The results are surprising; in particular, the largest exceptional number in this range is 343867. The author comments on certain theorems concerning representations of large integers as the sum of p tetrahedrals, and remarks on the gap between rigorous mathematical proofs and the empirical evidence.

L. Fox (Oxford)

COMPUTING MACHINES

See also 5727, 5737, 6229, 6242, 6313.

6195:

***Proceedings of the fifth annual computer applications symposium**, October 29-30, 1958. Sponsored by Armour Research Foundation of Illinois Institute of Technology, Technology Center, Chicago, Ill., 1959. x+153 pp. \$3.00.

The symposium discussed business and management applications on the first day, engineering and scientific application on the second. The emphasis was on the use of new computers and accessories, new techniques of computer programming, the organization and operation of computer installations, and new applications.

6196:

***Marek, Jindřich.** Maschinelle Interpolation bei Funktionen von zwei unabhängigen Veränderlichen. Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 83-86. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

"Es wird ein Interpolationsverfahren beschrieben, das

die Interpolation bei Funktionen zweier Veränderlicher aus von vornherein modifizierten Funktionstafeln wesentlich erleichtert. Die Methode wird im Zusammenhang mit der Verwendung von Lochkartenrechenmaschinen durchgeführt." (Zusammenfassung des Autors)

S. Gorn (Philadelphia, Pa.)

MECHANICS OF PARTICLES AND SYSTEMS

6197:

***McCuskey, S. W.** An introduction to advanced dynamics. Addison-Wesley Series in Mechanics. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959. viii+263 pp. \$8.50.

This textbook provides a simple treatment of selected topics from classical mechanics including the equations of Lagrange, Euler, Hamilton, and Hamilton-Jacobi. The material presented is appropriate for a one-semester course for students having some knowledge of elementary mechanics and advanced calculus.

P. Franklin (Cambridge, Mass.)

6198:

***Szebehely, V. G.** Orbit changes and invariants in a Newtonian central force field. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11-14, 1958, pp. 233-238. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. \$20.00.

The effect of a sudden change in velocity at one point on an orbit under the inverse-square law is studied. The new and old orbits are compared when the change leaves invariant the total energy, the angular momentum, or the direction of the velocity vector.

P. Franklin (Cambridge, Mass.)

6199:

Vălcovici, Victor. Sur les liaisons holonomes et non holonomes. *Acad. R. P. Romîne. Stud. Cerc. Mec. Apl.* 9 (1958), 811-818. (Romanian. Russian and French summaries)

"Considérant les liaisons comme une restriction des déplacements infinitésimaux, donc comme une condition fonctionnelle de ces grandeurs, l'auteur analyse les formes que peuvent prendre les liaisons dans cette conception. En admettant que l'équation fondamentale de la dynamique s'exprime — ainsi qu'il est universellement admis — par une forme linéaire des déplacements virtuels du système, égale à zéro, à l'aide du principe du travail virtuel, on aboutit à la conclusion que toute liaison sera exprimée sous une forme linéaire analogue, égale à zéro, les coefficients de cette forme étant des fonctions du temps, des vitesses et des accélérations. Il s'ensuit que la distinction que l'on fait d'habitude entre les liaisons holonomes et celles non holonomes est inopérante; la raison en est que les équations du mouvement s'écrivent dans tous les cas de la même façon, en utilisant, par exemple, les multiplicateurs de Lagrange." *Résumé de l'auteur*

6200:

***Athen, Hermann.** Ballistik. 2te Aufl. Quelle and Meyer, Heidelberg, 1958. 258 pp. DM 29.00.

This is a second edition, extensively revised, of a successful 1941 text. In brief compass the author touches upon all the major problems of classical ballistics, with considerable emphasis upon historical (and currently

obsolescent) aspects, and here brings the information fairly up to date. This text might disappoint anyone not finding ready access to the rich source material cited, since brevity forces the author to treat explicitly only the major aspects, leaving details to suggestive hints and references. Despite much recent material the author fails even to hint at much that has become standard practice (in America) for some two decades. Here the era of space-travel has not yet dawned.

A. A. Bennett (Providence, R.I.)

6201:

Turkovskii, V. A. On a brachistochrone in a field of constant force. *Ukrain. Mat. Ž.* 10 (1958), 336-339. (Russian)

6202:

Skowroński, Janisław; and Ziemba, Stefan. Some complementary remarks on the delta method for determining phase trajectories of systems with strong non-linearity. *Arch. Mech. Stos.* 10 (1958), 699-706. (Polish and Russian summaries)

The differential equation is of the form

$$\ddot{x} + \omega^2 x + F(x, \dot{x}, t) = 0,$$

which is equivalent to $\dot{x} = \omega^2 y$, $\dot{y} = -(x + \delta)$ with $\delta = \omega^{-2} F(x, \dot{x}, t)$. Starting in the phase plane at the initial point $P_0 = (x_0, y_0)$, the trajectory is approximated by an arc of the circle of angle $\omega \Delta t$ from $(-\delta_0, 0)$ to P_0 where $\delta_0 = \omega^{-2} F(x_0, \dot{x}_0, t_0)$. This locates P_1 and the process is iterated. Two methods of checking are proposed. In the case of forced oscillations a correction which is said to be sometimes useful is suggested.

J. P. LaSalle (Baltimore, Md.)

6203:

Bradistilov, G.; et Boyadjiev, G. Mouvements relatifs périodiques et asymptotiques de n -pendules physiques multiples dans un plan. *C. R. Acad. Bulgare Sci.* 10 (1957), 443-446. (Russian summary)

This article investigates periodic and asymptotic motions of a system of successively linked physical pendulums distributed in a vertical plane which is rotating uniformly about an axis in the plane. It is a continuation of the article by Bradistilov in same *C. R.* 8 (1955), no. 4, 5-8; listed in MR 19, 591.

6204:

Fracijs de Veubeke, B. Le problème du maximum de rayon d'action dans un champ de gravitation uniforme. *Astronaut. Acta* 4 (1958), 1-14.

"The problem of maximum range in a uniform gravitational field is treated, after a method indicated by Cicala [Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. 89 (1954-55), 350-358], as a Mayer problem of the calculus of variations. However, the use of time as independent variable has been discarded in favour of a parametric representation which facilitates the selection of a differential system ensuring the most compact numerical calculations; see the author's paper [Bull. Soc. Math. Belgique 8 (1956), 136-157; MR 21 # 994]. Various approximations are examined and particular attention is given to the ballistic extension of the trajectory and its junction with the propulsive phase. The use of classical criteria, such as that of Weierstrass, makes it possible to decide if the stationary solution provided by the Euler

equations gives a true maximum. However, the quantitative criterion of Jacobi cannot be verified otherwise than by numerical integration." *Author's summary*

6205:

Dumitrescu, Lucian; and Comănescu, Traian. Influence of the thrust regime on the performances of a space-ship taking-off tangentially from an artificial satellite. *Acad. R. P. Romîne. Stud. Cerc. Mec. Apl.* 9 (1958), 537-544. (Romanian. Russian and English summaries)

"The paper deals with the motion of a space rocket, taking-off from an artificial satellite, under the following assumptions: the thrust magnitude remains constant throughout the combustion time, and its direction remains tangential to the satellite orbit. The differential equations of the motion are solved approximatively by an iteration process.

From the authors' summary

STATISTICAL THERMODYNAMICS AND MECHANICS

6206:

Muckenfuss, Charles; and Curtiss, C. F. Kinetic theory of nonspherical molecules. III. *J. Chem. Phys.* 29 (1958), 1257-1272.

[For parts I, II see Curtiss, same *J.* 24 (1956), 224-241; Curtiss and Muckenfuss, *ibid.* 26 (1957), 1619-1636.]

The kinetic theory of multicomponent gases is generalised to cover nonspherical molecules. The distribution function depends now on the orientation and angular velocity of the molecules, as well as on their position and linear velocity. The Boltzmann equation for this function is obtained, and a consideration is given of the general equations of change for the various macroscopic physical quantities involved. The linearised Boltzmann equation is then solved by a variational procedure, similar to that of Chapman and Enskog, the transport coefficients being obtained in terms of certain integrals depending only on the geometry of the molecules, and carried out over the surfaces of two colliding molecules. These integrals are evaluated for a spherocylindrical model, and the results applied to a detailed consideration of self-diffusion.

S. Simons (London)

6207:

Gherman, O. Kinetic theory of the flow of a gas through a cylindrical tube. *Soviet Physics. JETP* 34(7) (1958), 1016-1019 (1470-1474 *Ž. Eksper. Teoret. Fiz.*).

By introducing a certain "anisotropy function" a good description of the flow of a gas can be obtained, even at pressures at which only empirical formulas have been used hitherto. Furthermore, the term corresponding to the slipping of the gas relative to the walls is obtained automatically, without any additional hypotheses; the same is true of the minimum rate of flow at intermediate pressures. Our final formula is qualitatively correct at all pressures, including intermediate ones.

Author's summary

6208:

Toda, Morikazu. On the theory of the Brownian motion. *J. Phys. Soc. Japan* 13 (1958), 1266-1280.

Brownian motion is treated as a special case of the general statistical mechanics developed by Prigogine and the author [*Molecular Phys.* 1 (1958), 48-62; MR 19, 1208].

D. ter Haar (Oxford)

ELASTICITY, PLASTICITY

6209:

Teleman, Sil'viu. La méthode de la projection orthogonale et les deux premiers problèmes de la théorie de l'élasticité. *Z. Čist. Prikl. Mat.* 1 (1956), 55-73. (Russian)

A Russian translation from the Romanian in Acad. R. P. Romine. *Bul. Ști. Sect. Ști. Mat. Fiz.* 7 (1955), 105-125 [MR 17, 684].

6210:

Cole, J.; and Huth, J. Stresses produced in a half plane by moving loads. *J. Appl. Mech.* 25 (1958), 433-436.

Displacements and stresses in an elastic half plane subjected to a concentrated load moving with constant speed along the straight boundary are studied. The problem is governed by two partial differential equations of second order corresponding to the dilatation wave and the distortion wave. The type of these equations depends upon the ratios between the elastic constants, the density of the material and the velocity of the applied load. Three different cases are discussed: (1) Both equations are elliptic (subsonic motion), in which case the author refers to results given previously by I. N. Sneddon [*Rend. Circ. Mat. Palermo* (2) 1 (1952), 57-62; MR 17, 802]; (2) one equation is elliptic, the other hyperbolic (transonic case); (3) both equations are hyperbolic (supersonic case). The appropriate techniques for the different types are applied.

W. Schumann (Zürich)

6211:

Loo, Tsu-tao. A second approximation solution on the elastic contact problem. *Sci. Sinica* 7 (1958), 1235-1246.

6212:

★Poritsky, H.; and Danforth, C. E. On the torsion problem. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11-14, 1958, pp. 431-441. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. \$20.00.

The authors briefly summarize St. Venant's theory of torsion for linear elastic materials, including definitions of the center of twist. They then attempt to shed light on errors resulting from using various approximate methods of solution. Among other things, the discussion adds further evidence that approximating functions by polynomials of comparatively low degree frequently leads to serious error.

J. L. Ericksen (Baltimore, Md.)

6213:

Kostandyan, B. A. Torsion of a shaft with affixed disk. *Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk* 11 (1958), no. 3, 63-77. (Russian. Armenian summary)

6214:

Gulkanyan, N. O. On torsion of prismatic rods of rectangular cross-section with asymmetrical rectangular cut. *Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk* 10 (1957), no. 5, 33-58. (Russian. Armenian summary)

6215:

Arčašnikov, V. P.; and Molyukov, I. D. On forms of stable semi-arches and arches. *Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh.* no. 6(10) (1957), 27-32. (Russian. Kazah summary)

The article examines the limiting equilibrium of arches

and semi-arches resulting from the weight of the arch itself. In the solution of the problem use is made of the parabolic condition of limiting equilibrium, which best describes the limiting state of rocks. [cf. Sokolovskii, V. V. *Prikl. Mat. Meh.* 20 (1956), 73-86; MR 18, 352].

Authors' summary

6216:

Kušul', M. Ya. The bending of cantilever plates bounded by piecewise smooth curves. *Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk* 1958, no. 10, 133-138. (Russian)

6217:

Minasyan, R. S. Torsion and bending of anisotropic, prismatic rods with parallelogram cross section. *Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk* 11 (1958), no. 3, 41-62. (Russian. Armenian summary)

6218:

Medwadowski, S. J. A refined theory of elastic, orthotropic plates. *J. Appl. Mech.* 25 (1958), 437-443.

"A refined theory of elastic, orthotropic plates is presented. The theory includes the effect of transverse shear deformation and normal stress and may be considered a generalization of the classical theory of von Kármán modified by the refinements of the Levy-Reissner-Mindlin theories. A nonlinear system of equations is derived directly from the corresponding equations of the three-dimensional theory of elasticity in which body-force terms have been retained. Next, the system of equations is linearized and reduced to a single sixth-order partial differential equation in a stress function. A Levy-type solution of this equation is discussed." (From the author's summary)

L. S. D. Morley (Farnborough)

6219:

Gol'denveizer, A. L. Equations of the theory of shells in terms of displacements and stress functions. *Prikl. Mat. Meh.* 21 (1957), 801-814. (Russian)

The fundamental relations of thin shell theory are developed in terms of an arbitrary system of curvilinear coordinates (in general not orthogonal) associated with the middle surface. Tensor notation is used. The article is closely related to earlier work of the author [same *Prikl.* 9 (1945), 463-478; MR 7, 351].

R. C. T. Smith (Armidale)

6220:

DeSilva, C. Nevin; and Naghdi, P. M. Asymptotic solutions of a class of elastic shells of revolution with variable thickness. *Quart. Appl. Math.* 15 (1957), 169-182.

Seit den ersten bahnbrechenden Arbeiten von H. Reissner und E. Meissner über die achsensymmetrische Biegung der Rotationsschalen für kleine Verformungen, haben sich viele Untersuchungen mit diesem Gegenstand beschäftigt, wobei aber in der Mehrzahl aller Fälle eine unveränderliche Schalenstärke vorausgesetzt wurde. Die beiden Verfasser konnten in einer früheren Arbeit zeigen, daß man die beiden simultanen reellen Differentialgleichungen, wie sie E. Reissner im Rahmen seiner neuen Theorie der Rotationsschalen 1949 aufgestellt hat, unter gewissen Voraussetzungen auf eine einzige komplexe Differentialgleichung zweiter Ordnung reduzieren kann, auf die sich das bekannte Langersche Verfahren der asymptotischen Integration anwenden läßt.

Die oben erwähnten Voraussetzungen bestehen in der Konstanz eines gewissen Faktors, der von den geometrischen Größen der Schale und ihrer Dicke sowie deren Ab-

leitungen abhängt. Man kann diese Forderung als eine Differentialgleichung zweiter Ordnung für die Schalensstärke deuten, deren verschiedene Lösungen die Verfasser in der vorliegenden Schrift studieren. Darunter findet sich auch ein Fall, der früher schon von E. Meissner auf anderem Wege behandelt wurde. Den Abschluß bildet eine Anwendung auf gewisse flache Schalenformen, wo sich besonders weitgehende Vereinfachungen ergeben.

W. Zerna (Hannover)

6221:

Ambartsumian, S. A. On a general theory of anisotropic shells. J. Appl. Math. Mech. 22 (1958), 305-319 (226-237 Prikl. Mat. Meh.).

In der vorliegenden Arbeit stellt der Verfasser eine in mehrfacher Hinsicht recht interessante Schalentheorie auf. Bezüglich des Materialverhaltens geht er nicht von dem einfachen Hookeschen Gesetz für den isotropen Fall aus, sondern setzt eine allgemeine, wenn auch noch lineare Abhängigkeit der Spannungen von den Verzerungen voraus, wobei er lediglich an der Symmetrie des Formänderungsverhaltens bezüglich der Schalenmittelfläche festhält. Bezüglich der Formänderungen wird in üblicher Weise angenommen, daß die Normalspannungen in den Schnitten parallel zur Mittelfläche keinen nennenswerten Einfluß ausüben, und daß die Länge der Linien-elemente senkrecht zur Mittelfläche ungeändert bleibt. Andererseits behält der Verfasser die Normalenhypothese in ihrer üblichen Form nicht bei, sondern ersetzt sie durch eine Aussage über die Verteilung der Schubspannungen über die Schalendicke. Hier nimmt er in Anlehnung an die bekannte Reissnersche Plattentheorie einen parabolischen Verlauf an, eine Hypothese, die schon früher von Green und Zerna in die Theorie der isotropen Schalen eingeführt wurde.

Unter Verwendung der Hauptkrümmungslinien als Koordinatennetz und der Wlassow'schen Bezeichnungsweise entwickelt der Verfasser sodann die naturgemäß recht umständlichen Formeln seiner Theorie und stellt Randbedingungen für die verschiedensten Fälle der Stützung auf. Im Gegensatz zur üblichen Schalentheorie können an jedem Rand fünf Bedingungen befriedigt werden.

Zum Abschluß wendet der Verfasser seine Formeln auf den Fall der Kreiszylinderschale an, wobei er die Fälle der vollkommenen und der nur transversalen Isotropie behandelt. Durch Vergleichsrechnungen mit der üblichen Theorie gibt er den durch die Normalenhypothese bedingten Fehler im ersten Fall zu ca. 15% an, während er im letzteren Fall je nach den Verhältnissen bis auf 35% anwachsen kann.

W. Zerna (Hannover)

6222:

Choudhury, P. On bending of a circular plate of anisotropic material under certain non-uniform distribution of load. Indian J. Theoret. Phys. 5 (1957), 97-104.

"In this paper, a solution of the fundamental equation for an elastic material with transverse isotropy has been obtained in the form of polynomials and it has been used in discussing the problem of bending of a circular plate under non-uniformly distributed load, the intensity of which follows a parabolic law of distribution."

Author's summary

6223:

Vekua, I. N. Conditions guaranteeing a momentless strain state of equilibrium of a convex shell. Soobšč. Akad. Nauk Gruz. SSR 20 (1958), 525-532. (Russian)

6224:

*Chapkin, R. L.; and Williams, M. L. Stress singularities for a sharp-notched polarly orthotropic plate. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11-14, 1958, pp. 281-286. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. \$20.00.

Generalizing a series of earlier investigations by M. L. Williams, the authors consider the nature of possible bending and extensional stress singularities at the corner of a sector-shaped thin plate with polar orthotropy. The pole of the orthotropy is assumed coincident with the corner of the plate. The characteristic equations appropriate to the usual combinations of boundary conditions are derived. The case of free-free extension (or clamped-clamped bending) of a plate with a semi-infinite line crack, i.e., a notch of vanishing opening angle, is examined in some detail.

E. Sternberg (Providence, R.I.)

6225a:

Bassali, W. A.; and Nassif, M. A thin circular plate normally and uniformly loaded over a concentric elliptic patch. Proc. Cambridge Philos. Soc. 55 (1959), 101-109.

6225b:

Bassali, W. A. Bending of a thin circular plate under hydrostatic pressure over a concentric ellipse. Proc. Cambridge Philos. Soc. 55 (1959), 110-120.

The solution is obtained, for the usual restraints at the circular boundary, by using complex variables. Explicit formulae are given for the moments and shears on the boundary and at the centre of the plate.

L. S. D. Morley (Farnborough)

6226:

Filonenko, G. G. On the motion of a wheel rolling on an elastic rail. Akad. Nauk Ukrain. RSR. Prikl. Meh. 4 (1958), 182-191. (Ukrainian. Russian and English summaries)

"The problem of the motion of a wheel with a spring-supported mass on an elastic rail is reduced to the solution of a system of differential equations with variable coefficients.

In the present paper the author gives a solution of this system of differential equations and investigates the nature of motion of a wheel with a spring-supported mass on an elastic rail in the first field of parametric resonance. It is shown that in this field the interaction of wheel and rail is discontinuous."

From the author's summary

6227:

Kaul, R. K.; and Tewari, S. G. On the bounds of eigenvalues of a clamped plate in tension. J. Appl. Mech. 25 (1958), 52-56.

This paper utilizes Kato's theorem and Temple's generalization to obtain a lower bound to the first eigenvalue of a clamped rectangular plate in uniform biaxial tension. The principal difficulty is the determination of the elastic mode corresponding to the "residual forces" in the approximation of the first eigenvalue. The authors have solved this problem by an auxiliary variational method and obtained upper and lower bounds which differ by less than 0.004%. The high accuracy of the lower bound was unexpected and is surprising.

G. Temple (Oxford)

6228:

Huffington, N. J., Jr.; and Hoppmann, W. H., II. On the transverse vibrations of rectangular orthotropic plates. *J. Appl. Mech.* 25 (1958), 389-395.

"Frequency equations and eigenfunctions are obtained for the flexural vibrations of rectangular plates of orthotropic material, having two axes of elastic symmetry in the plane of the plate and parallel to its edges, for various boundary conditions, all having two parallel edges simply supported. The method used is based on that of M. Lévy [C. R. Acad. Sci. Paris 129 (1899), 535-539] for static problems. For a broader class of boundary-value problems, orthogonality criteria and expressions for kinetic and potential energies are derived; these are of value in dealing with forced vibrations." (From the authors' summary) G. B. Warburton (Edinburgh)

6229:

★Atkinson, Cyril P.; and Bourne, Charles P. The solution of Duffing's equation for the softening spring system using the Ritz-Galerkin method with a three term approximation. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11-14, 1958, pp. 71-77. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. \$20.00.

"This paper presents the results of an investigation of the solutions of the differential equation (Duffing's equation) $\ddot{q} + q - 4q^3/3 = F \cos \omega t$ based on a three term approximation to the solution $q(t)$. The assumed solution is of the form $q(t) = A \cos \omega t + B \cos 3\omega t + C \cos 5\omega t$. The method of solution used is known as the Ritz averaging method (or the Ritz-Galerkin method). The differential equation was solved for both free and forced oscillations. Solutions are presented in the form of amplitude vs frequency plots for the three components of the assumed solution. Resonance of the harmonics is indicated in certain regions of the amplitude vs frequency plane. In these regions the fundamental component alone is not adequate in describing the response of the system." (Author's summary) M. Lister (State College, Pa.)

6230:

Kellenberger, W. Biegeschwingungen einer unrunder, rotierenden Welle in horizontaler Lage. *Ing.-Arch.* 26 (1958), 302-318.

The paper considers the flexural vibrations of a shaft of non-circular cross-section, supported in bearings at its ends and rotating about the horizontal axis through the bearings. By expressing displacements relative to rotating axes, coincident with the principal axes of the cross-section, the system can be represented by two differential equations with constant coefficients. First the frequency equation is derived from the homogeneous equations; the occurrence of unstable roots is studied in terms of a 'lack of roundness' parameter. The inhomogeneous equations contain additional terms due to gravity and to the eccentricity of the mass centre relative to the geometric centre of the cross-section; the forced vibrations resulting from these two effects are considered separately. G. B. Warburton (Edinburgh)

6231:

Vorovič, I. I. Certain problems of shell stability in the large. *Dokl. Akad. Nauk SSSR* 122 (1958), 37-40. (Russian)

Statements without proofs of results such as: the load at which more than one position of equilibrium becomes

possible, according to the non-linear theory of curved shells, is less than the buckling load deduced from the linearized equations. R. C. T. Smith (Armidale)

6232:

Predeleanu, M. Über die Verschiebungsfunktionen für das achsensymmetrische Problem der Elastodynamik. *Z. Angew. Math. Mech.* 38 (1958), 402-405.

6233:

Buchwald, V. T. Transverse waves in elastic plates. *Quart. J. Mech. Appl. Math.* 11 (1958), 498-508.

"Expressions are derived for the phase and group velocities of plane rotational waves in elastic plates with displacement vector parallel to the surfaces of the plates and perpendicular to the direction of propagation. It is shown that in a double plate two types of transverse wave can travel along the plate, each type consisting of an infinite number of modes. The behaviour of the waves is illustrated by dispersion curves, computed for the lowest modes. The paper concludes with an investigation of the properties of Love waves in a surface layer, including a discussion of the behaviour of the minimum group velocity, discovered by Jeffreys, for which a lower bound is found." (From the author's summary)

L. S. D. Morley (Farnborough)

6234:

Lockett, F. J. Effect of thermal properties of a solid on the velocity of Rayleigh waves. *J. Mech. Phys. Solids* 7 (1958), 71-75.

"This note is concerned with the propagation of Rayleigh waves in an isotropic thermoelastic solid. It is found that, within the frequency range normally attainable, the velocity of propagation of these waves can be determined from the classical equation merely by replacing the parameter $\beta^2 = (\lambda + 2\mu)/\mu$ occurring in that equation by $(1 + \epsilon)\beta^2$, where $\epsilon = \gamma^2 v_p^2 T(1 + \nu)^2/c(1 - \nu)^2$, γ is the coefficient of linear expansion, v_p is the velocity of purely elastic longitudinal waves in the solid, T is the absolute temperature of the solid in its reference state of uniformly zero stress and strain, c is the specific heat at constant strain and ν is Poisson's ratio. A typical case is examined numerically and it is found that taking into account the thermal properties of the solid produces a difference of less than one per cent in the velocity and amplitude of the Rayleigh waves." (From author's summary)

L. S. D. Morley (Farnborough)

6235:

Paria, Gunadhar. Love waves in hypoelastic body of grade zero. *Quart. J. Mech. Appl. Math.* 11 (1958), 509-512.

The author finds that the wave velocity of Love waves in a hypoelastic body of grade zero retains the same character as that in the classical linearly elastic body. One of the normal stresses is, however, not identically zero, whereas all the normal stresses are zero in the classical elastic body. A. E. Green (Newcastle-upon-Tyne)

6236:

Przemieniecki, J. S. Thermal stresses in rectangular plates. *Aero. Quart.* 10 (1959), 65-78.

The author obtains an approximate series solution for the thermal stresses in a rectangular plate which is free from loading and is subjected to a given temperature distribution that is uniform across the plate thickness. The problem is treated within classical two-dimensional elasticity theory by expanding the Laplacian of the

temperature distribution as well as the Airy function in terms of the characteristic functions appropriate to the transverse vibrations of a clamped-damped rectangular plate. The prescribed temperature field T is thus restricted by the requirement that $\nabla^2 T$ and the normal derivative of $\nabla^2 T$ vanish along all four edges. A numerical example is worked out in detail.

E. Sternberg (Providence, R.I.)

6237:

*Goodier, J. N. Formulas for overall thermoelastic deformation. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11-14, 1958, pp. 343-345. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. \$20.00.

On the basis of the Betti reciprocal theorem generalized to thermoelastic problems, formulas are found for volume change of an arbitrary body and of a cavity, as well as for mean extension, flexural rotation, terminal deflection and torsional rotation of a bar.

J. Nowinski (Madison, Wis.)

6238:

Derski, Włodzimierz. On transient thermal stresses in a thin circular plate. Arch. Mech. Stos. 10 (1958), 551-558. (Polish and Russian summaries)

Transient thermal axially symmetric state of stress in a circular plate suddenly heated at the rim is investigated. No loss of heat or loss according to Newton's law is postulated. Thermoelastic potential of displacement Φ is used, and the solution is composed of two parts: one depending on Φ , and the other pertaining to the athermal problem.

J. Nowinski (Madison, Wis.)

6239:

Trostel, R. Stationäre Wärmespannungen mit temperaturabhängigen Stoffwerten. Ing.-Arch. 26 (1958), 416-434.

General displacement field equations are derived, in the case of an arbitrary steady state of temperature T , for an isotropic elastic body with temperature dependent Young's modulus E and coefficient of thermal expansion, but constant Poisson's ratio. Expanding the solution, in the sense of Poincaré, in a series with respect to a parameter which includes the material constants of the law $E=E(T)$, one gets a set of displacement field equations and stress tensors corresponding to the consecutive terms of the series. The proper boundary-value problem is satisfied by the first partial solution (basic term of the series), the boundary conditions for the additional solutions being homogeneous. Plane state of stress is discussed and two problems are solved, concerning a thick-walled cylinder undergoing an axially symmetrical temperature gradient, and a uniform beam in pure bending and stretching, with unsymmetrical cross-section, the temperature varying normally to the axis of the beam in the plane of symmetry. A numerical example for a steel double-tee beam is solved.

The reviewer finds the method of solution ingenious.

J. Nowinski (Madison, Wis.)

6240:

Trostel, R. Wärmespannungen in Hohlzylindern mit temperaturabhängigen Stoffwerten. Ing.-Arch. 26 (1958), 134-142.

Thermal stresses in hollow circular cylinder are investigated assuming elastic incompressibility of the material and temperature dependent Young's modulus E ,

coefficient of thermal expansion α and conductivity k . Simple general formulas in closed form for any axially symmetrical temperature field T and any variation of E , α and k with T are obtained. It is shown that, for the assumed temperature gradient, constant k instead of variable k can be adopted. Two numerical examples involving linear variability of E and α with temperature are solved for different thermal boundary conditions.

J. Nowinski (Madison, Wis.)

6241:

Mossakowska, Zofia; and Nowacki, Witold. Thermal stresses in transversally isotropic bodies. Arch. Mech. Stos. 10 (1958), 569-603. (Polish and Russian summaries)

STRUCTURE OF MATTER

6242:

Ahmed, F. R.; and Barnes, W. H. Generalized programmes for crystallographic computations. Acta Cryst. 11 (1958), 669-671.

Details are given of a number of crystallographic programs written for the Ferranti computer FERUT. These programs carry out Fourier syntheses, differential syntheses, structure factor calculations and determinations of the accuracy of atomic coordinates and thermal parameters. In all relevant programs isotropic thermal vibration is assumed.

The programs are written in such a way that maximum use is made of the space group symmetry of the crystal analysed. This is accomplished by writing the structure factor as a sum of products of three trigonometric functions. For example, in the triclinic, monoclinic, and orthorhombic systems the structure factor has the form:

$$A, B(hkl) =$$

$$C \left[\sum_r [f_r(hkl) \times \left[\sum \cos 2\pi h(x) \cdot \cos 2\pi k(y) \cdot \cos 2\pi l(z) \right] \right],$$

where $A(hkl)$, $B(hkl)$ are, respectively, the real and the imaginary parts of the structure factor for the plane (hkl) , $f(hkl)$ is the thermally corrected scattering factor for the r th atom in the plane (hkl) , and x, y, z are the coordinates of the r th atom.

A pseudo code, using a number of constants, is introduced with the data to reduce the general expression to the specific expression for the space group involved.

Included in the output of the program are the quantities $\sum (\Delta F)^2$, $\sum h^2 (\Delta F)^2$, $\sum k^2 (\Delta F)^2$, $\sum l^2 (\Delta F)^2$, where $\Delta F = F_{\text{obs}} - F_{\text{calc}}$. These quantities are used in estimating the standard deviations of the atomic coordinates and the electron density distribution.

W. M. Macintyre (Boulder, Colo.)

6243:

Hauptman, H.; and Karle, J. Seminvariants for centrosymmetric space groups with conventional centered cells. Acta Cryst. 12 (1959), 93-97.

The nature of the dependence of phase on the choice of origin is clarified for those centrosymmetric space groups for which the conventional unit cell is not primitive, by means of special linear combinations of the phases and the structure semi-invariants. The theory leads to simple procedures for selecting the origin by first fixing the functional form for the structure factor and then specifying arbitrarily the values of a suitable set of phases.

Werner Nowacki (Bern)

6244:

Wentzel, Gregor. Meissner effect. Phys. Rev. (2) 111 (1958), 1488-1492.

Starting from the Bardeen-Cooper-Schrieffer model of a superconductor [same Rev. 108 (1957), 1175-1204; MR 20 #2196] the existence of a Meissner effect is derived by a method which is gauge invariant in every step. In evaluating the Meissner effect, use is made of Bogoliubov's transformation [Nuovo Cimento (10) 7 (1958), 794-805]. The present treatment predicts a Meissner effect with a penetration depth of the magnetic field which is larger than found in the Bardeen-Cooper-Schrieffer paper. The present paper has been criticized by Pines and Schrieffer [Phys. Rev. Lett. 1 (1958), 407-408]. Also Rickayzen [Phys. Rev. 111 (1958), 817-821] finds a penetration depth of the magnetic field which is at variance with the one found in this paper. Wentzel, however, pointed out in a rebuttal [Phys. Rev. Lett. 2 (1959), 33-34] that the treatment given in the presently reviewed paper is correct and preferable to Rickayzen's approach and also to the not strictly gauge invariant treatment by Bardeen-Cooper-Schrieffer.

H. Statz (Waltham, Mass.)

FLUID MECHANICS, ACOUSTICS

See also 6275, 6317, 6318, 6319.

6245:

★Comolet, R. Introduction à l'analyse dimensionnelle et aux problèmes de similitude en mécanique des fluides. Masson et Cie, Paris, 1958. 116 pp. 1600 fr.

Ce livre fort bien présenté et magnifiquement illustré s'adresse surtout aux ingénieurs. Après l'énoncé des principaux résultats de l'analyse dimensionnelle, l'auteur étudie les similitudes des systèmes rencontrés en mécanique des fluides et plus spécialement en hydraulique.

P. Germain (Paris)

6246:

Pohsraryan, M. S.; and Sanoyan, V. G. Hydrodynamical calculation of a plane flow with a lateral outlet. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk 10 (1957), no. 6, 25-40. (Russian. Armenian summary)

A free flow in a channel with an outlet exhibits a stream separation line (the streamline function there is zero). The form of the separation line, and the flow in the outlet and in the channel past the outlet are determined. The effects of the ratio of the cross section area of the outlet to that of the channel are taken into account. Classical equations are used, and conformal mapping is applied for the solution of the problem. Theoretical results agree, within 7 percent, with the experimental results.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

6247:

Ivanova, L. S. Impact of a liquid on the inclined wall of an infinite, partly closed container. J. Appl. Math. Mech. 22 (1958), 344-348 (254-256 Prikl. Mat. Meh.).

In this paper the following plane problem is discussed. A semi-infinite container of finite depth is filled with incompressible non-viscous liquid. The boundaries of the container consist of a horizontal base, an inclined wall, and a horizontal top plate adjacent to the wall of finite length. The container suddenly acquires a horizontal velocity, and the pressure on the wall is determined by means of complex variable methods. The solution can

be applied to the study of the impact of liquid on an inclined dam or the impact of liquid cargo on a bulkhead of a ship.

R. M. Morris (Cardiff)

6248:

Kapur, J. N. Transverse component of velocity in a plane symmetrical jet of a compressible fluid. Quart. J. Mech. Appl. Math. 11 (1958), 423-426.

In this note, the author corrects an error which appeared in a paper by D. G. Toose [same J. 5 (1952), 155-164; MR 13, 1002].

Y. H. Kuo (Peking)

6249:

Sideriades, Lefteri. Cheminées d'équilibre: étude de l'amortissement des oscillations; problème des maxima et des minima. C. R. Acad. Sci. Paris 247 (1958), 1171-1173.

The differential equations are of the form

$$\frac{h}{g} \frac{dW}{dt} + Z + P_0 W^2 = 0,$$

$$\left((W - F) \frac{dZ}{dt} \right) (Z + Z_0 + P_0 W^2) = K,$$

where W is related to the rate of flow and Z to the height of water in the channel. For various values of K the singular point is a stable focus and lies on the parabola $Z + P_0 W^2 = 0$. A relation is given between the maximum and minimum values of Z as the solutions spiral toward the focus.

J. P. LaSalle (Baltimore, Md.)

6250:

Sideriades, Lefteri. Oscillateur hydraulique à relaxation basé sur le principe des cheminées d'équilibre. C. R. Acad. Sci. Paris 247 (1958), 1296-1299.

It is shown by a phase plane study of the solutions that for some channel shapes as in the case of a triode oscillation, the nonlinear system can have a single stable limit cycle.

J. P. LaSalle (Baltimore, Md.)

6251:

Sideriades, Lefteri. De l'influence des formes de l'insertion sur la stabilité des cheminées d'équilibre. C. R. Acad. Sci. Paris 247 (1958), 1089-1092.

A problem in the flow of fluid through a channel gives rise to a nonlinear system $\dot{x} = y$, $\dot{y} = Q/P$, where P and Q are polynomials in x and y with a coefficient having one value for $y > 0$ and another value for $y < 0$. A sufficient condition for stability of a singular point is obtained from linear approximations in the two half-planes. The condition requires the solutions of this linear approximation about the singular point to spiral in.

J. P. LaSalle (Baltimore, Md.)

6252:

Roseau, Maurice. Short waves parallel to the shore over a sloping beach. Comm. Pure Appl. Math. 11 (1958), 433-493.

The equations of edge waves are considered: The function $\phi(x, y)$ is defined in the sector $0 < x < \infty$, $-x \tan \alpha < y < 0$ (where α is an acute angle), in which it satisfies $(\Delta - h^2)\phi = 0$. The boundary conditions are $\phi_y - \phi = 0$ for $y = 0$, $x > 0$; $\phi_x \sin \alpha + \phi_y \cos \alpha = 0$ for $y \cos \alpha + x \sin \alpha = 0$, $x > 0$. The author looks for those explicit solutions with singularities at the origin which can be represented as Laplace integrals. The solution of the equations is then reduced to the solution of a functional equation in the complex plane. Most of the paper is concerned with the

case $k > 1$ (short waves); if θ is defined by $\cos \theta = k^{-1}$, there are two solutions ϕ_1 and ϕ_2 dying out at infinity, and with the following behaviour near the origin: i) when $\pi - 2\theta = 2n\alpha$, then $\phi_1 = 0$ and ϕ_2 is regular at 0 when n is an odd integer (Stokes edge wave and known generalizations); ϕ_1 has a logarithmic singularity at 0 and $\phi_2 = 0$ when n is an even integer; (ii) when $\pi - 2\theta \neq 2n\alpha$, a) if $\alpha = \pi/2m$, where m is an integer, then $\phi_2 = 0$ and ϕ_1 behaves like $r^{-\pi/\alpha}$ near 0; b) if $3\pi - 2\theta = (2l+1)2\alpha$, where l is an integer, then $\phi_2 \cos(\pi^2/\alpha) - \phi_1 \sin(\pi^2/\alpha) = 0$, and ϕ_1 and ϕ_2 both behave like $r^{-\pi/\alpha}$ near 0; c) if $\alpha \neq \pi/2m$ and $3\pi - 2\theta \neq (2l+1)2\alpha$, then $\phi_2 \cos(\pi^2/\alpha) - \phi_1 \sin(\pi^2/\alpha)$ has a logarithmic singularity at 0.

A construction is also given for an infinite sequence of solutions $\phi_n^{(p)}$ ($p=1, 2, 3, \dots$), small at infinity and behaving like $r^{-\pi/\alpha}$ near 0. A similar sequence can be defined when $0 \leq k < 1$ and the wave at infinity is prescribed. The treatment is elegant, as in previous work of this author, but more remains to be done on this problem. In particular, the reviewer wonders whether all solutions of the system, bounded at infinity, might be represented in the postulated form? *F. Ursell* (Cambridge, England)

6253:

Barta, Stefan. Note on the derivation of Laplace's equation on surface tension. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 8 (1958), 123-126. (Slovak)

6254:

Kaškarov, V. P. Jet motion of a viscous fluid in a conical diffusor. *Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. no. 6(10)* (1957), 20-26. (Russian. Kazah summary)

An exact solution of the equations of viscous fluid flow for axially symmetric motion in a conical diffusor is obtained. Related problems were treated by H. B. Squire [*Quart. J. Mech. Appl. Math.* 4 (1951), 321-329; MR 13, 294]. The method used in present paper is based on Squire's approach.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

6255:

Emersleben, O. Wie hängt bei Parallelströmung zäher Flüssigkeiten die Durchflussmenge von der Gestalt des Querschnitts ab? *Wiss. Z. Univ. Greifswald. Math.-Nat. Reihe* 6 (1956/57), 321-339.

6256:

Jenson, V. G. Viscous flow round a sphere at low Reynolds numbers (< 40). *Proc. Roy. Soc. London. Ser. A* 249 (1959), 346-366.

The Navier-Stokes equation is solved, in terms of stream function and vorticity, in spherical polar co-ordination, with $\log r$ used as radial variable to introduce conveniently the proper grading of mesh size in the finite-difference equations. The latter are treated by relaxation methods, each partial solution for one function giving starting values for the other. It is not clear whether the convergence of this process was automatic or forced by the "short cuts learned by experience", an important point in view of the experience of other authors. Tables and pictures are given for solutions with Reynolds numbers 5, 10, 20 and 40, and compared favourably with previous results. *L. Fox* (Oxford)

6257:

Săvulescu, Șt. N. Considérations sur certaines solutions pour le cas de la couche limite compressible non permanente. *Acad. R. P. Române. Stud. Cerc. Mec. Apl.* 9 (1958), 867-879. (Romanian. Russian and French summaries)

"L'auteur établit les conditions que doivent remplir les paramètres extérieurs de la couche limite compressible non permanente (vitesse du courant extérieur, température du courant extérieur et température à la paroi) afin de pouvoir étendre la solution de Crocco ($U_a = E_a$) à ce cas. On donne également les conditions pour lesquelles $U_a(\eta) = E_a(\eta)$, où $\eta = \psi(t, x, y)/\psi_0(t, x)$.

A l'aide de la méthode des profils type de vitesses et de températures, on examine certains cas particuliers pour lesquels on détermine les formules du frottement et du transfert de chaleur, dans le cas du nombre de Prandtl unitaire. On constate des différences intéressantes par rapport à l'écoulement permanent." *Résumé de l'auteur*

6258:

le Fur, Bernard. Calcul de la couche limite laminaire dans un écoulement compressible avec gradient de pression sur une paroi thermiquement isolée. *J. Rech. Centre Nat. Rech. Sci.* 42 (1958), 9-18.

6259:

Ludford, G. S. S.; and Schot, S. H. Sonic limit singularities. I. General theory. II. Examples. *Arch. Rational Mech. Anal.* 2 (1958), 160-172; 173-190.

This paper constitutes an extremely full review of what can be said about conditions under which solutions of Chaplygin's equation for steady two-dimensional homentropic gas-flows, which are regular in the hodograph plane, possess singular curves (where the mapping onto the physical plane is singular) which include sonic points. *M. J. Lighthill* (Manchester)

6260:

Quilghini, Demore. Sull'equilibrio spontaneo di una massa fluida soggetta alla propria gravitazione. *Riv. Mat. Univ. Parma* 8 (1957), 27-41.

The author determines the conditions which must be imposed on the function $p = p(\rho)$ so that an isolated (zero pressure at the surface) barotropic fluid may assume the spherical configuration of equilibrium under its own gravitation. A sufficient condition is $p(\rho) < (5/6)\rho(d p(\rho)/d \rho)$. The results are extended to a van de Waals gas and to gases stratified in spherical layers.

C. D. Calsoyas (Livermore, Calif.)

6261:

Naylor, D. Degenerate waves in unsteady gas flow. *J. Math. Mech.* 7 (1958), 705-722.

Consider isentropic potential flows with velocity $(u, v, w) = q = q(\lambda, \mu)$ and speed of sound $a = a(\lambda, \mu, t)$, where $\lambda = \lambda(r, t)$, $\mu = \mu(r, t)$, and r is the coordinate vector. The principal technique for investigation of such flows is a succession of Legendre transformations. In particular, by means of $\Phi = \phi - r \cdot q$, where ϕ is the velocity potential function, one can show that q is constant on a two-parameter family of straight lines. For the parametrization $\lambda = u$, $\mu = v$, $w = w(u, v)$ the partial differential equation for ϕ implies, as the principal possibility, that $w = w(u, v)$ is a developable surface and, furthermore, that Φ and w must satisfy two partial differential equations. By a transformation of variables too involved to describe here it is shown that one of these equations

implies that $\psi = t\phi/\partial t - \Phi$ satisfies an Euler-Poisson equation, characteristic variables for which are generalizations of the Riemann invariants of one-dimensional flow. The other equation connecting Φ and w is so complicated that the author has not been able to determine its implications. However, he has shown that there are no flows such that at each instant t all of the lines $\lambda = \text{const.}$, $\mu = \text{const.}$ pass through a common point $r_0(t)$.

J. H. Giese (Aberdeen, Md.)

6262:

Tyler, Edmund F. The oscillating inboard flap at supersonic speeds. *J. Aero./Space Sci.* 26 (1959), 56-58.

6263:

Goldsworthy, F. A. The structure of a contact region, with application to the reflexion of a shock from a heat-conducting wall. *J. Fluid Mech.* 5 (1959), 164-176.

The flow near a temperature discontinuity in the one-dimensional motion of a compressible gas is studied. It is shown that the pressure across this region is approximately constant, and the temperature distribution is found for three different functional relations between the thermal conductivity and temperature.

As an example the contact region behind a shock reflected from a plane wall is studied. By matching the flow in the contact region with the inviscid flow outside it the author is able to estimate the attenuation of the shock.

K. Stewartson (Durham)

6264:

Kanevskii, I. N.; and Rozenberg, L. D. Calculation of the sonic field in the focal region of a cylindrical focusing system. *Akust. Zh.* 3 (1957), 46-61. (Russian)

The sound field due to a circular-cylindrical radiator with areally constant excitation of arbitrary opening is calculated, with particular attention to the region about the focal axis, for the asymptotic case $2kR\alpha \gg 1$ (R —radius, 2α —angular opening), that is, under neglect of marginal diffraction. The treatment is conventional, comprising all cases admitting closed analytic solutions as well as Bessel function expansions for the focal and steepest descent integration for the intermediate region in the general case. Extent of the focal region as defined by the first zeros in the focal plane and the first minima along the normal are discussed in detail; this includes consideration of the effect of truncation of the cylinder down to a length/diameter ratio of 1.

H. G. Baerwald (Albuquerque, N.M.)

6265:

Criminale, William O., Jr.; Ericksen, J. L.; and Filbey, G. L., Jr. Steady shear flow of non-Newtonian fluids. *Arch. Rational Mech. Anal.* 1 (1958), 410-417.

According to the linear theory of viscous fluids, it is always possible for a fluid flowing through a cylindrical tube, to which it adheres, to undergo steady rectilinear motion, each particle moving with constant speed in a straight line parallel to the generators of the cylinder. Recent theoretical work has shown that, for non-Newtonian fluids, this simple type of motion is possible only for very special shapes of tubes and that, in general, it is replaced by a flow consisting of such a rectilinear motion combined with a secondary flow in cross-sectional planes. Analysis given by present authors for a class of incompressible non-Newtonian fluids indicates that, if the conclusions which Roberts [Proc. Second Intern. Rheology Congress, Oxford, 1953, pp. 91-98, Academic Press, New York, 1954] drew from his experiments are correct, the fluids which he observed should be capable of under-

going rectilinear motion through tubes of any shape. Markovitz [Trans. Soc. Rheol. 1 (1957), 37-52] draws conclusions from other experimental data which contradict some of Robert's conclusions. Calculations made by the authors in accordance with these conclusions suggest that secondary flows are to be expected in some shapes of tubes. In both cases the experimental data referred to is mainly for polyisobutylene solutions and may not be representative for other fluids. A partial proof suggests that if secondary flows do not occur in elliptical tubes, they will not occur in tubes of other shapes.

A. E. Green (Newcastle-upon-Tyne)

6266:

Simon, René. On the reflection and refraction of hydromagnetic waves at the boundary of two compressible gaseous media. *Astrophys. J.* 128 (1958), 392-397.

The author discusses the reflection and refraction of a plane hydromagnetic wave between two semi-infinite homogeneous compressible media of infinite electrical conductivity. A detailed analysis of the laws of reflection and refraction is given for the case of a uniform magnetic field directed perpendicular to the interface. In general, three types of reflected waves and three types of refracted waves are possible.

H. Greenspan (Cambridge, Mass.)

6267:

Pilatovskii, V. P. Propagation of fluctuations along the boundary of separation when inhomogeneous flow in porous media is formed by relative sliding of fluids. *Ukrain. Mat. Zh.* 10 (1958), 280-288. (Russian. English summary)

6268:

Zautykov, O. A. On the solution of a filtration problem. *Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh.* no. 6(10) (1957), 46-50. (Russian. Kazah summary)

6269:

Pilatovskii, V. P. Investigation of the stability of a homogeneous filtration flow on a thin conical bed. *Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk* 1958, no. 11, 43-49. (Russian)

6270:

Lyaško, I. I. Determination of the issuing velocities of filtrations under a multiple-groove dike in the presence of a curvilinear underground water-support. *Kiev. Derzh. Univ. Nauk Zap.* 16 (1957), no. 2=Kiev. Gos. Univ. Mat. Sb. 9 (1957), 99-110. (Russian)

6271:

Oleinik, O. A.; Kalašnikov, A. S.; and Čžou, Yui-Lin'. The Cauchy problem and boundary problems for equations of the type of non-stationary filtration. *Izv. Akad. Nauk SSSR. Ser. Mat.* 22 (1958), 667-704. (Russian)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 6165.

6272:

*Слюсарен, Г. Г. О возможном и невозможном в оптике. [Slyusarev, G. G. On the possible and the impossible in optics.] 2nd ed., revised. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 178 pp. 2.90 rubles.

The opening chapter, "Burnt from a distance", opens

with the story of the burning of the ships in the harbor at Syracuse, and proceeds to consider the possibility in general of noxious rays. The conclusion is that sources 10^4 or 10^5 times as powerful as those presently available would be required.

The remaining chapters have less drama, but are interesting and informative. The headings are: Optical fables; reversibility and irreversibility in optics; limits of resolution in optical systems; phase, amplitude and image.

A. S. Householder (Oak Ridge, Tenn.)

6273:

Low, F. E. A Lagrangian formulation of the Boltzmann-Vlasov equation for plasmas. *Proc. Roy. Soc. London, Ser. A* 248 (1958), 282-287.

A variational principle is given from which the Boltzmann-Vlasov equation for an ionized gas in an electromagnetic field may be derived by varying the particle paths and the four-vector potential of the electromagnetic field. This principle is used to give a new formulation of the problem of small oscillations about equilibrium. No new calculations on the latter problem are reported in the paper.

A. H. Taub (Urbana, Ill.)

6274:

*Ландау, Л. Д.; и Лифшиц Е. М. Электродинамика сплошных сред. [Landau L. D.; and Lifshic, E. M. *Electrodynamics of continuous media*.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 532 pp. 11.70 rubles.

The series of texts on theoretical physics by the authors has been announced for publication in English translation in 9 volumes. The present volume, on one of the older fields of physics, promises to be one of the most successful from a pedagogic point of view.

The subject matter is that usually known as "macroscopic electrodynamics". However, only about one-half of the volume is devoted to subject matter traditionally put under this title. The authors have drawn on their extensive knowledge of modern physics to introduce new material usually treated in other contexts.

Electrostatics of conductors and dielectrics, fields with constant currents, magnetostatics, ferromagnetism, and quasi-stationary fields take up about one-half of the volume. The treatment is traditional, for the most part, the novelty consisting mainly in the selection of material and the careful discussion. Two brief chapters on superconductivity and magnetohydrodynamics will be welcomed by many. Then come three chapters on electromagnetic wave propagation and theoretical optics. The final chapters cover the transmission of fast particles through matter, electromagnetic fluctuation, scattering of electromagnetic waves, and the diffraction of X-rays in crystals.

The treatment is nowhere very deep, from the point of view of the expert, but it will provide the student with a good summary of the part of the subject which can be rationalized in the form of self-contained mathematical problems. It will appeal to the mathematical reader who wishes to know something of how theoretical physicists think, but who does not really care whether the phenomena referred to exist in actual fact.

Quite a number of examples, in the form of problems with solutions, appear throughout the text. No problems for solution by the student are given. This deviation from current practice in American and English texts may lead to some difficulty in class room use, unless problems are included in the translated edition.

E. L. Hill (Minneapolis, Minn.)

6275:

Cooke, J. C. The coaxial circular disc problem. *Z. Angew. Math. Mech.* 38 (1958), 349-356. (German, French and Russian summaries)

"Two problems involving equal coaxial circular discs are discussed, (i) the discs charged to equal, or equal and opposite, potentials, and (ii) the discs rotating slowly with equal, or equal and opposite, angular velocities in a viscous fluid. In this paper some improved values are given for the cases of most interest, namely the "condensor problem" and the "viscometer problem", that is to say, the cases of opposite potentials or directions of rotation. When the discs are close together the numerical work is very laborious and a semi-empirical method is given whereby important constants can be obtained as a correction to an approximation originally due to Maxwell, which is simple numerically. An attempt is also made to assess the effect of placing the disc system inside a coaxial cylinder." (Author's summary)

A. E. Heins (Pittsburgh, Pa.)

6276:

Schmidt, Helmut. Eine kausale Theorie für Wellen, die sich mit Überlichtgeschwindigkeit ausbreiten. *Z. Physik* 151 (1958), 408-420.

6277:

Butcher, A. C.; and Lowndes, J. S. The diffraction of transient electro-magnetic waves by a wedge. *Proc. Edinburgh Math. Soc.* 11 (1958/59), 95-103.

A representation of the field due to a line source of current, which is an arbitrary function of time, lying parallel to the edge of a perfectly conducting infinite wedge is obtained by straightforward application of Laplace transforms and Fourier series. For the special case of step-function time dependence and a medium with short relaxation time, it is shown that the result may be interpreted in terms of images and shadow regions.

E. T. Kornhauser (Providence, R.I.)

6278:

Hauser, Walter. On the theory of anisotropic obstacles in waveguides. *Quart. J. Mech. Appl. Math.* 11 (1958), 427-437.

It is recommended that a method based on Schwinger's variational principle be used to compute the scattering matrix of an anisotropic obstacle in a waveguide.

C. H. Papas (Pasadena, Calif.)

6279:

Chirlian, Paul. Bounds on the error in the unit step response of a network. *Quart. Appl. Math.* 16 (1958), 432-435.

The transient response of a network can be obtained from the network response to unit step excitation. Very often some network characteristic, such as the transfer function, may only approximate the desired behavior, and it is of interest to determine how this approximation may affect the transient response. Two rather general departures of network transfer functions from the specified characteristics are treated, (1) the band-limiting approximation, and (2) deviations of amplitude or phase functions. Integral expressions are developed by means of which bounds may be placed upon the error in the unit step response of a network for the approximations stated above.

R. Kahal (Washington, D.C.)

6280:

Sternberg, Robert L.; Shipman, Jerome S.; and Zohn, Shirley Rose. Multiple Fourier analysis in rectifier problems. *Quart. Appl. Math.* 16 (1958), 335-360.

The general problem considered here is concerned with the results of a multiple Fourier analysis of the output from a cut-off power law rectifier responding to a several frequency input under various assumptions about the cut-off bias, the exponent of the power law, and input amplitude ratios. In particular, this paper deals with the evaluation of the single, double, or triple integrals in the multiple Fourier series expansion of the rectifier output corresponding to one, two, or three frequency inputs, respectively. In general, the integrals representing the Fourier coefficients cannot be evaluated in closed form in terms of elementary functions. However, solutions are obtained in various types of transcendental forms. Hypergeometric representations and power series expansions for these coefficients and the recurrence relations satisfied by them are given. Graphs of the first ten basic functions for the single frequency input problem and of the first fifteen basic functions for the two-frequency input problem are also included in the paper. Computations for the average output power when the input frequency ratios are all irrational, that is, when the combined input is non-periodic, are also investigated.

R. Kahal (Washington, D.C.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 6263.

6281:

García Moliner, F. On irreversible entropy production. *An. Real Soc. Españ. Fis. Quim. Ser. A* 55 (1959), 33-36. (Spanish. English summary)

The explicit contribution of an arbitrary type of momentum-dependent force to irreversible entropy production is shown to vanish for forces \mathbf{F} such that $\nabla_{\mathbf{k}} \cdot \mathbf{F} = 0$, $\nabla_{\mathbf{k}}$ being the gradient operator in \mathbf{k} -space. The relation to previous treatments of some such cases is illustrated. Some thermodynamical implications are discussed.

Author's summary

6282:

Frössling, Nils. Calculation by series expansion of the aerodynamic heating at laminar constant-property boundary layers. *Lunds Univ. Årsskr. N.F. Avd. 2* 54 (1958) = *Kungl. Fysiogr. Sällsk. Handl. N.F.* 69 (1958), no. 9, 17 pp.

The paper extends the author's previous investigation [same *Årsskr.* 36 (1940), no. 4; English translation, *NACA TM* 1432, 1958; *MR* 2, 331] on the heat transfer problem of a laminar flow past a blunt body by including the viscous heating. The study is restricted to constant fluid properties and adiabatic wall temperature (zero temperature gradient at the wall). Using the "similarity" variables of the boundary layer theory, the energy equation is solved by series expansions. The two dimensional problems are studied with a free stream flow being (a) a series of odd powers of x and (b) a series of all powers. The axisymmetric flow is limited to (c) that of odd powers. Numerical calculations with Prandtl number $= 0.7$ are given for cases (a) and (c). A comparison with previous experiments is made. L. N. Tao (Chicago, Ill.)

QUANTUM MECHANICS

6283:

Fano, U.; and Racah, G. Irreducible tensorial sets. *Pure and Applied Physics*. Vol. 4. Academic Press Inc., New York, 1959. vii+171 pp. \$6.80.

The coupling and recoupling of angular momenta occupies an increasingly important position in the theories of nuclear and atomic structure and in theories of nuclear and atomic scattering. The underlying mathematical methods may be described in terms of the theory of group representations or in terms of the algebra of tensor operators, and the latter approach is adopted in the book under review. Defining a tensorial set to consist of sets of quantities (such as states of atomic systems and tensor components) with common transformation properties and an irreducible tensorial set to be a set that cannot be resolved into subsets corresponding to separate linear transformations, the authors construct an algebra of considerable generality and power, which serves to unify the different approaches that have been developed earlier. The concepts governing the construction of the algebra are clearly and carefully discussed, the detailed properties being relegated to a number of appendices. The second half of the book is concerned with applications of the algebra to some quantum problems of current importance, and the great utility of the methods is demonstrated. The book is not useful as a working manual since it contains few formulas and no tables, but it is essential reading for those interested in the fundamentals of coupling problems in quantum mechanics.

A. Dalgarno (Belfast)

6284:

Landé, Alfred. γ superposition and quantum periodicity. *Phys. Rev. (2)* 108 (1957), 891-893.

An effort is made to base quantum mechanics on a set of postulates which are, in turn, motivated by requirements on the relation between probabilities and probability amplitudes.

H. W. Lewis (Madison, Wis.)

6285:

Martin, J. L. The evaluation of matrix elements in functional integral form. *Proc. Roy. Soc. London. Ser. A* 248 (1958), 560-567.

The source function technique of Schwinger [*Proc. Nat. Acad. Sci. U.S.A.* 37 (1951), 452-459; *MR* 13, 520] is used to express quantum mechanical matrix elements in terms of functional-derivative operators acting on certain standard functionals. These expressions are then transformed using the algorithm

$$M\left(\frac{\delta}{\delta x}\right)N(x)|_{x=0}=N\left(\frac{\delta}{\delta x}\right)M(x)|_{x=0},$$

where M and N are any two functionals. For transition amplitudes relating to coordinate eigenstates the result is a particular form of the Feynman principle integral, but the author's method is also applicable to other matrix elements including those relating to eigenstates of operators with discrete eigenvalues.

J. C. Polkinghorne (Cambridge, England)

6286:

Seaton, M. J. The quantum defect method. *Monthly Not. Roy. Astr. Soc.* 118 (1958), 504-518.

The quantum defect method uses interpolated or extrapolated quantum defects to determine asymptotic forms

for atomic wave functions, both for positive energies and for negative energies other than the eigenvalues. This paper presents a summary of previous work on the mathematical processes involved for the case of a central field tending to the attractive Coulomb form at large distances. Some new results for positive energy states and on the normalisation of bound state wave functions are also given with numerical values. There are a discussion of the application to calculation of accurate phase factors, elastic and inelastic collision cross sections, and a number of references to earlier work. *D. F. Mayers (Oxford)*

6287:

Goldberg, Irwin. Gauge-invariant quantum electrodynamics. *Phys. Rev. (2)* 112 (1958), 1361-1366.

The author formally completes the construction of quantum electrodynamics in terms of observables only by means of methods developed earlier [P. G. Bergmann and I. Goldberg, *Phys. Rev. (2)* 98 (1955), 531-538; MR 17, 565; P. G. Bergmann, *Nuovo Cimento (10)* 3 (1956), 1177-1185; MR 18, 540]. The development does not include consideration of divergence difficulties or renormalization procedures. Attention should be called to refinements due to P. A. M. Dirac [*Canad. J. Phys.* 33 (1955), 650-660; MR 17, 926]. *P. G. Bergmann (New York, N.Y.)*

6288:

Fried, H. M.; and Yennie, D. R. New techniques in the Lamb shift calculation. *Phys. Rev. (2)* 112 (1958), 1391-1404.

"Based on the analogy between the calculations of radiative corrections to electron scattering and the Lamb shift, a new procedure applicable to bound state self-energy problems is developed, wherein the electron propagator is expanded in powers of the external potential. In the resulting sequence, a change in the gauge of the virtual photon field conveniently removes from each term spurious lower order contributions and yields a new and considerably simpler sequence; the expectation value of the first two terms of the latter is shown to account for the major portion of the Bethe logarithm. A simple method is developed to sum all the $a(Z\alpha)^4\mu$ dependence, and the result is the lowest order Lamb shift formula. The ease of the calculation, as well as that involved in obtaining the relativistic level shift correction of order $a(Z\alpha)^6\mu$ (not given in the present paper), indicates that the method may find application in the calculation of further higher order effects." (Author's summary)

F. Rohrlich (Baltimore, Md.)

6289:

Redmond, P. J. Elimination of ghosts in propagators. *Phys. Rev. (2)* 112 (1958), 1404-1408.

This paper exhibits an approximation to the (meson) propagator, Δ_F' , with the following properties. (1) It meets the analyticity requirements of Lehmann. (2) It differs from the Feldman propagator by having the "ghost" pole subtracted. (3) As a function of g^2 it has an essential singularity at $g^2=0$. (4) Its asymptotic expansion in g^2 agrees with perturbation theory. (5) The corresponding value of Z is finite and satisfies $0 < Z < 1$. *J. C. Taylor (London)*

6290:

Scarf, Frederick L. Scattering amplitudes for the Thirring model. *Phys. Rev. (2)* 111 (1958), 1433-1435.

The Thirring model is an exactly soluble relativistic field theory of a self-interacting particle in two dimensions (one space and one time). The author writes

down the lowest order covariant equation for the two-particle scattering amplitude, and solves the equation for the amplitude. Spurious solutions to the equation appear, which are not related to the exact expression for the two-body amplitude. The point is that an exact solution to an approximate equation is to be regarded with due caution. *H. W. Lewis (Madison, Wis.)*

6291:

Werle, J. Phenomenological theory of the S-matrix and T, C and P invariance. *Nuovo Cimento (10)* 9 (1958), 569-585.

The S-matrix elements for any process due to the interaction of 4 bosons, of 2 bosons and 2 fermions, and of 4 fermions, are tensors under proper isochronous Lorentz transformations. Additional symmetry properties are obtained by the requirements of hermiticity or of invariance under space-inversion (P), particle conjugation (C), time-reversal (T), or PCT. These properties are independent of any approximations or specific assumptions concerning the interaction. A proof of the PCT theorem for these processes is also given.

F. Rohrlich (Baltimore, Md.)

6292:

Werle, J. Phenomenological theory of the S-matrix and T, C and P invariance. II. *Nuovo Cimento (10)* 9 (1958), 1091-1106.

The paper contains some applications of the general phenomenological theory of the transition matrix which has been given in part I of this work. [Cf. preceding review]. Besides a general discussion of various physical relations which follow from the T, C and TCP invariance requirements the examples of $K_{\mu 3}$ and hyperon decays are studied in some more detail. (Author's summary)

F. Rohrlich (Baltimore, Md.)

6293:

Lüders, Gerhart. Proof of the TCP theorem. *Ann. Physics* 2 (1957), 1-15.

This paper contains a simple proof of the important TCP theorem, which is central to present discussions of the validity of local field theories. The author, who originally discovered the theorem [G. Lüders, *Danske Vid. Selsk. Mat.-Fys. Medd.* 28 (1954), no. 5; MR 16, 315; W. Pauli, "Niels Bohr and the development of physics", McGraw-Hill, New York, and Pergamon Press, London, 1955; MR 17, 692] gives explicitly the transformations of field quantities generated by T, C and P, and then, following Pauli, demonstrates that the product TCP may be split into the product of two operations — strong reflection and Hermitian conjugation. The latter two operations separately preserve the field equations and commutation relations in interaction representation, and, in virtue of the fact that the interaction Hamiltonian density is a Hermitian operator which is the 00 component of a tensor, the Schrodinger equation is also invariant. Hence any local field theory is invariant under the product of these two operations. *C. A. Hurst (Adelaide)*

6294:

Glaser, V. An explicit solution of the Thirring model. *Nuovo Cimento (10)* 9 (1958), 990-1006.

The Thirring model is a quantized relativistic field theory of a spin $\frac{1}{2}$ field with interaction Lagrangian $g\bar{\psi}\psi\psi$ and with only one space dimension. [*Ann. Physics* 3 (1958), 91-112; MR 19, 1016]. The author succeeds in giving a complete solution of the field equations. In the calculation of the eigenvectors of the energy-momentum

operator an ambiguity arises: Depending on the limiting process by which the direct interaction (involving a δ -function) is obtained, one obtains two sets of eigenvectors which differ in their functional dependence on the coupling constant g . After renormalization by the (diverging) constant $Z^{\frac{1}{2}}$ the matrix elements of the field operators are finite analytic functions of g . The theory is trivial in the sense that the S -matrix is diagonal in the physical particle representation: elastic scattering simply means a phase change, and there is no particle creation.

F. Rohrlich (Baltimore, Md.)

6295:

Thirring, W. On interacting spinor fields in one dimension. *Nuovo Cimento* (10) 9 (1958), 1007-1015.

In the light of Glaser's solution (see preceding review) the physical significance of the Thirring model and its relation to other models, especially the Lee model, are discussed. The theory describes particles of zero mass going in the positive or negative direction of the single space dimension. The charges Q_1 , Q_2 going in each direction are constants of motion corresponding to the invariance under the two parameter transformation group $\psi \rightarrow \exp[i(\alpha + \beta \gamma_5)]\psi$. The exclusion principle also strongly restricts the possible processes. There is a divergent wave function renormalization, but no charge renormalization, no ghost states, and no need for negative probabilities.

F. Rohrlich (Baltimore, Md.)

6296:

Haag, R. Quantum field theories with composite particles and asymptotic conditions. *Phys. Rev.* (2) 112 (1958), 669-673.

This paper provides a clear and concise discussion of the asymptotic conditions in a quantum field theory where there is a many-to-one correspondence between particles and fields. It exhibits the connection between the methods of Nishijima and Zimmermann [K. Nishijima, *Phys. Rev.* 111 (1958), 995-1011; MR 20#3007; W. Zimmermann, *Nuovo Cimento* (10) 10 (1958), 597-614; MR 21#588] and of Ekstein [H. Ekstein, *Nuovo Cimento* (10) 4 (1956), 1017-1058; MR 18, 626], called by the author the methods of weak and strong convergence, respectively.

The model discussed is the theory of a single scalar field $A(x)$ describing two spinless particles of masses m , M and satisfying the usual conditions (covariance, absence of states with negative energy, local commutativity, existence of an equation of motion). Incoming and outgoing fields for each particle are introduced in terms of their physical interpretation, e.g.,

$$\Phi^{\text{in}}(x) = \sum_{\alpha} f_{\alpha}(x) u_{\alpha}^{\text{in}} + f_{\alpha}^{*}(x) u_{\alpha}^{\text{in}\dagger},$$

where u_{α}^{in} annihilates an incoming M -particle from a state of wave function $f_{\alpha}(x)$ belonging to a complete orthonormal set of positive energy solutions of the Klein-Gordon equation $(\square - M^2)f(x) = 0$; similar expressions define Φ^{out} and the m -particle fields ϕ^{in} , ϕ^{out} . Given any operator field $C(x)$ and a Klein-Gordon wave function $f(x)$, one may construct

$$C_f(t) = i \int_{x_0=t} (C \partial / \partial x_0 - f \partial C / \partial x_0) d^3x.$$

The author proves two theorems. (I) Weak convergence. Let $B(x)$ be any "almost-local operator field", defined as the limit of a polynomial in the basic field $A(x)$,

$$B(x) = h^{(0)} + \int h^{(1)}(x-y) A(y) dy \\ + \int h^{(2)}(x-y_1, x-y_2) A(y_1) A(y_2) dy_1 dy_2 + \dots,$$

where the coefficient functions $h^{(n)}$ are sufficiently smooth and decrease sufficiently rapidly as any of the points y_i move away from x . Choose the coefficients so that $\langle 0|B(x)|0\rangle = 0$, $\langle \alpha|B(x)|0\rangle \neq 0$, where $|0\rangle$ is the vacuum and $|\alpha\rangle$ is a one M -particle state. Then a mass- M wave function $g(x)$ may be found such that

$$\lim_{t \rightarrow -\infty} \langle \Psi_1 | B_g(t) | \Psi_2 \rangle = \langle \Psi_1 | u_{\alpha}^{\text{in}\dagger} | \Psi_2 \rangle$$

for any normalizable states Ψ_1 , Ψ_2 . (II) Strong convergence. If $Q(x)$ is any almost-local field which satisfies the conditions of theorem (I) and if, further, $Q(x)|0\rangle$ is a one M -particle state, then one may find a mass- M wave function $g(x)$ such that

$$\lim_{t \rightarrow -\infty} \|Q_g(t)|\Psi\rangle - u_{\alpha}^{\text{in}\dagger}|\Psi\rangle\| = 0$$

for any normalizable Ψ .

The same theorems are valid for the m -particle operators, with mass- m wave functions replacing those of mass M . Throughout the discussion there is a complete symmetry in the treatment of the two particles: neither can be regarded as more elementary than the other. The only distinction which may be made arises in cases where there is a selection rule.

P. W. Higgs (London)

6297:

Scherrer, Willy. Zur linearen Feldtheorie. V. (Ein asymmetrisches Wirkungsprinzip). *Z. Physik* 152 (1958), 319-327.

[For parts I-IV, see same *Z.* 139 (1954), 44-55; 140 (1955), 164-180, 374-385; 144 (1956), 373-387; MR 16, 635, 756; 17, 305, 909.] The author discusses field equations derived from a Lagrangean which involves linear combinations of six invariants previously [ibid. 138 (1954), 16-34; 139 (1954), 44-55; MR 16, 79, 635] discussed. These invariants involve a constant 4×4 symmetric matrix and a constant 4×4 antisymmetric matrix of very special form. In this paper the various conditions are derived which must be satisfied by the coefficients of the invariants in order that the field equations obtained from the variational principle reduce to the Einstein field equations for the vacuum. In addition, the field equations obtained when the Lagrangean contains so-called "anti-symmetric" invariants are discussed.

A. H. Taub (Urbana, Ill.)

6298:

Brown, Laurie M. Two-component Fermion theory. *Phys. Rev.* (2) 111 (1958), 957-964.

The equations of motion for a charged particle of spin $\frac{1}{2}$ in interaction with a (quantized) electromagnetic field are formulated in terms of differential equations of second order. Implicit use of a supplementary condition (in identifying two spinor functions) reduces the number of independent complex functions from 8 to 4, which can be chosen so as to satisfy the Dirac equation. This method fails when an anomalous moment coupling with the electromagnetic field is included, which is taken as an argument to exclude such a coupling. Rules for calculation of matrix elements of the spin $\frac{1}{2}$ particle interacting with a quantized electromagnetic field are developed which are particularly simple. The present theory does not apply to quantized spinor fields, at least as developed in this paper.

E. C. G. Sudarshan (Cambridge, Mass.)

6299:

Frahn, W. E.; and Lemmer, R. H. Non-static effects on individual nucleons in a spheroidal potential. *Nuovo Cimento* (10) 6 (1957), 664-673.

The authors consider the effect on shell-model calculations of using a non-local potential to describe the interaction between a nucleon and the nucleus. The non-locality is assumed to be of very short range, and this range is used as an expansion parameter. By taking terms of second order in the range they derive a modified wave equation. If the range is assumed constant (as a function of energy), the effective mass of the nucleon is given as a function of the depth of the local (shell-model) potential. When the local potential is taken to be a spheroidal oscillator potential, the main effect of the non-locality of the potential is to increase the deformation of the nucleus by an amount which is estimated to be 25%.

D. J. Thouless (Birmingham)

6300:

Meckler, Alvin. Majorana formula. *Phys. Rev.* (2) 111 (1958), 1447-1449.

The author gives another derivation of a formula originally derived by Majorana which gives the probability of a spin transition from a state of magnetic quantum number m to one of magnetic quantum number m' . The derivation does not compound the general case out of the spin $-\frac{1}{2}$ case but directly follows the dynamics of the spin vector.

A. H. Taub (Urbana, Ill.)

6301:

Lüders, Gerhart. Zum Teilchen-Loch-Übergang in Systemen nahezu unabhängiger Fermi-Teilchen. *Z. Naturf.* 14a (1959), 5-7.

There exists a well-known isomorphism in the description of a system of quasi-independent fermions, whereby expectation values and transition rates can be calculated by regarding the system as consisting either of particles or holes. The author realizes this correspondence as an antilinear relationship. This is claimed to be more aesthetic than the conventional Heisenberg antisymmetrization.

S. Bludman (Berkeley, Calif.)

6302a:

Fujii, Kanji; and Iwata, Kenzo. The phenomenological model of the interaction of elementary particles. *Progr. Theoret. Phys.* 19 (1958), 475-484.

6302b:

Fujii, Kanji; and Iwata, Kenzō. Phenomenological model on the interactions of elementary particles. II. *Progr. Theoret. Phys.* 20 (1958), 126-132.

The first of these papers is an application of the principle of individual γ_5 transformation of fermion fields [B. Stech and J. H. D. Jensen, *Z. Physik* 141 (1955), 175-184, 403-410; J. Tiomno, *Nuovo Cimento* (10) 1 (1955), 226-232; MR 16, 1184] and the law of conservation of leptons [E. J. Konopinski and M. H. Mahmoud, *Phys. Rev.* 92 (1953), 1045-1049] to the characterisation of the interaction of elementary particles. All elementary particles are assumed to be fermions and all interactions are assumed to be four-fermion couplings. Parity violation in all weak interactions automatically follows. While several possible Hamiltonians governing weak interactions consistent with these restrictions are discussed, no decisive choice is made in this paper.

In the second paper it is claimed that the $V-A$ interaction is "a natural and almost unique consequence of our previously mentioned standpoint." But this choice from amongst the various alternatives of the first paper is, of course, based on new experimental evidence, though the authors assert that previous investigators who have made the same choice [R. P. Feynman and M. Gell-Mann, *Phys. Rev.* (2) 109 (1958), 193-198; MR 19, 813; J. J. Sakurai, *Nuovo Cimento* (10) 7 (1958), 649-660; E. C. G. Sudarshan and R. E. Marshak, *Phys. Rev.* 109 (1958), 1860-1861] have made "assumptions which are more or less ad hoc. The authors seem to be unaware of the comprehensive analysis of the problem by the reviewer and R. E. Marshak [Proceedings of the Conference on Mesons and Newly discovered Particles, Padua-Venice September 1957]. E. C. G. Sudarshan (Cambridge, Mass.)

6303:

Lüders, Gerhart. Zur Rolle lokalisierter Zustände der Bethe-Goldstone-Gleichung. *Z. Naturf.* 14a (1959), 1-5.

Using the Goldstone approach to the many-body problem, a discussion is given of the danger of the localized solutions of the Bethe-Goldstone equation.

D. ter Haar (Oxford)

RELATIVITY

See also 6314.

6304:

Larish, E.; and Shekhtman, I. Propagation of detonation waves in the presence of a magnetic field. *Soviet Physics. JETP* 35(8) (1959), 139-143 (203-207 *Z. Eksper. Teoret. Fiz.*).

It is shown that relativistic detonation waves in a magnetic field possess properties similar to those of the ordinary waves. Solutions of the equations at the discontinuity are presented for the relativistic and non-relativistic cases."

Authors' summary

6305:

Penrose, R. The apparent shape of a relativistically moving sphere. *Proc. Cambridge Philos. Soc.* 55 (1959), 137-139.

According to the special theory of relativity an object should appear to be flattened in the direction of motion. An instantaneous photograph of a rapidly moving sphere is, however, shown to have the same outline as that of a stationary sphere. Supposing the sphere to be at rest and the observer moving, the author derives the result by three independent methods by using in turn, (i) the relativistic aberration formula, (ii) a purely geometrical (space-time) argument and (iii) the properties of two-component spinors.

G. L. Clark (London)

6306:

Onicescu, O. La mécanique du solide rigide. *Acad. R. P. Romine. Stud. Cerc. Mec. Apl.* 9 (1958), 519-524. (Romanian. Russian and French summaries)

"Dans l'ensemble des principes introduits par l'auteur, on examine d'abord la mécanique du solide d'inertie et l'on détermine les impulsions d'inertie et l'énergie H , qui est la somme de l'énergie d'inertie mr^2 (m =masse relativiste) du centre de masse et de l'énergie de rotation.

On considère ensuite le champ déterminé par un po-

tentiel scalaire et un potentiel vecteur; on détermine les équations du mouvement. Dans le cas où le potentiel vecteur est nul, les équations se réduisent à celles de la mécanique classique (lorsque la vitesse de translation aussi est petite).

D'autres forces sont introduites par des relations de liaison".

Résumé de l'auteur

6307:

Bel, Louis. Sur la radiation gravitationnelle. C. R. Acad. Sci. Paris 247 (1958), 1094-1096.

If

$$T_{\beta\gamma\mu\nu} = \frac{1}{2} g_{\beta\gamma} g_{\mu\nu} R^{\rho\sigma}{}_{\rho\sigma} - (R^{\alpha}{}_{\beta}{}^{\lambda}{}_{\mu} R_{\alpha\gamma}{}^{\lambda}{}_{\nu} + R^{\alpha}{}_{\beta}{}^{\lambda}{}_{\nu} R_{\alpha\gamma}{}^{\lambda}{}_{\mu}),$$

then $T^{\beta}{}_{\gamma\mu\nu} = 0$, and $T_{00,00} = -2V$, where V is positive (unless $R_{\alpha\beta}{}^{\lambda}{}_{\lambda} = 0$, when $V = 0$). Further, from $T_{\beta\gamma\mu\nu}$ and an orthogonal ennuple can be defined a three-dimensional vector \mathbf{P} such that $T_{\beta\gamma\mu\nu}$, V and \mathbf{P} play the part of the Maxwell tensor, the energy density and the Poynting vector in gravitational radiation.

C. W. Kilmister (London)

6308:

Bel, Louis. Étude algébrique d'un certain type de tenseurs de courbure. Le cas 3 de Petrov. C. R. Acad. Sci. Paris 247 (1958), 2096-2099.

The author continues his geometrical study of the Riemann tensor in empty space-time [cf. #6307, see also J. Geheniau, C. R. Acad. Sci. Paris 244 (1957), 723-724; MR 19, 169; and earlier work referred to there]. He shows that in empty space-time the equations

$$R_{\alpha\beta\lambda\mu} l^{\alpha} l^{\lambda} = 0, \quad \eta_{\alpha\beta\gamma\delta} R_{\lambda\mu}{}^{\gamma\delta} l^{\alpha} l^{\lambda} = 0$$

(where $R_{\alpha\beta\lambda\mu}$ is the Riemann tensor and $\eta_{\alpha\beta\gamma\delta}$ is the alternating tensor) have a solution l^{α} (which must be a null vector) if and only if the Riemann tensor is of Petrov type II null or III. He shows that in the latter case, l^{α} must be a field of geodesic tangent vectors.

F. A. E. Pirani (Chapel Hill, N.C.)

6309:

Bel, Louis. Définition d'une densité d'énergie et d'un état de radiation totale généralisée. C. R. Acad. Sci. Paris 246 (1958), 3015-3018.

In a Riemannian space V_N of indefinite signature, it may happen that all the independent rational integral scalar concomitants (i.r.i.s.c.) formed from a given set $\{T, g\}$ of tensors, together with the metric tensor g_{mn} , may vanish, although not all of the tensors T themselves vanish. Example: in the hyperbolic normal space-time V_4 of general relativity, the only i.r.i.s.c. of a null vector $K^m = 0$ is $g_{mn} K^m K^n = 0$.

However, if the space V_N is sectioned into subspaces of definite signature by introducing suitable additional vectors A^m , then the vanishing of all the i.r.i.s.c. of $\{T, A, g\}$ (naturally excluding those formed from the A^m without any of the T 's) is necessary and sufficient for the vanishing of all the tensors T . In the above example: let A^m be an arbitrary timelike vector. If $g_{mn} K^m K^n = 0$ and $g_{mn} K^m A^n = 0$, then $K^m = 0$. The need for sectioning arises only because some of the tensors T may, as in this example, lie in special relation to the null cone of the V_N .

In particular, it may happen that in a space-time all the 14 i.r.i.s.c. of the Riemann tensor vanish, without the space-time necessarily being flat. It is for this reason that Synge, in constructing an "Invariant gravitational density" [Proc. Roy. Irish Acad. Sect. A. 58 (1957), 29-39; MR 19, 1140] found it necessary to introduce an arbitrary auxiliary timelike vector field in order to state the

condition that space-time be flat in terms of the vanishing of a scalar formed from the Riemann tensor.

In the paper under review, the author employs an auxiliary vector field to derive a condition similar to Synge's condition, but in a more geometrical way, and (in the reviewer's opinion) with less justification for calling it an energy density. He then proceeds to define "un état de radiation totale généralisée" which generalizes that proposed by Lichnerowicz [C. R. Acad. Sci. Paris 246 (1958), 893-896; MR 19, 1237].

F. A. E. Pirani (Chapel Hill, N.C.)

6310:

Newman, D. J.; and Kilmister, C. W. A new expression for Einstein's law of gravitation. Proc. Cambridge Philos. Soc. 55 (1959), 139-141.

The metric tensor is formed from vectors \mathcal{E}_μ which are based on the algebra of Eddington's E -numbers. It is shown that equations involving the \mathcal{E}_μ and their second order derivatives are equivalent to Einstein's equations $R_{\alpha\beta}$.

C. Gilbert (Newcastle-upon-Tyne)

6311:

Abrol, M. L.; and Mishra, R. S. On the field equations of Bonnor's unified theory. Tensor (N.S.) 8 (1958), 14-20.

The authors study the field equations of Bonnor's unified field theory, [W. B. Bonnor: Proc. Roy. Soc. London Ser. A 226 (1954), 366-377; MR 16, 755] under projective changes of affine connections.

M. Wyman (Edmonton, Alta.)

6312:

Souriau, Jean-Marie. Une axiomatique relativiste pour la microphysique. C. R. Acad. Sci. Paris 247 (1958), 1559-1562.

A five-dimensional theory is proposed with the single topological condition that the universe is homeomorphic to a "tube". Successive approximations correspond to general relativity, the theory of Jordan-Thiry and a theory bearing a formal resemblance to quantum mechanics.

C. W. Kilmister (London)

ASTRONOMY

See also 6260.

6313:

Gontkovskaya, V. T. The application of modern computational techniques to the analytical methods of celestial mechanics. Byull. Inst. Teoret. Astr. 6 (1958), 592-629. (Russian. English summary)

"The present paper deals with the application of punched-card machines and electronic digital computers to analytical methods of celestial mechanics. Four different methods are discussed, every one being estimated from the point of view of application both kinds of machines."

Author's summary

6314:

Brumberg, V. A. Les corrections relativistes dans la théorie de la lune. Byull. Inst. Teoret. Astr. 6 (1958), 733-756. (Russian. French summary)

"On obtient les corrections principales relativistes (en coordonnées harmoniques) dans la théorie de la lune en appliquant la méthode de Hill et Brown aux équations du mouvement du problème restreint dans la seconde approximation de la théorie de la relativité générale."

Résumé de l'auteur

6315:

Deberdeev, A. A. Bestimmung der störenden Kräfte aus den Bewegungen der Himmelskörper. Byull. Inst. Teoret. Astr. 6 (1958), 581-591. (Russian. German summary)

"Im vorliegenden Artikel wird eine Methode zur Bestimmung der auf einen Himmelskörper störend wirkenden Kräfte vorgeschlagen, wenn die Bewegung des letzteren bekannt ist.

"Um das mehrfache Differenzieren zu vermeiden, wird die Ermittlung der obengenannten Kräfte auf die Auflösung gewisser Integralgleichungen zurückgeführt, womit ein Ersetzen des angenäherten Differenzierens durch das angenäherte Integrieren — eine zuverlässigere Operation — ermöglicht wird."

Zusammenfassung des Autors

6316:

Bonnor, W. B. Stability of polytropic gas spheres. Monthly Not. Roy. Astr. Soc. 118 (1958), 523-527.

A polytrope of uniform index n is considered. This polytrope is perturbed in the following way: a concentric sphere S_0 of radius r_0 inside the polytrope suffers small changes of r_0 in such a way that the mass inside S_0 does not change, that the material inside S_0 remains in hydrostatic equilibrium and that the polytropic relation applies within S_0 . For small variations of the pressure and specific volume on S_0 , the equilibrium is stable if $1 < n \leq 3$. There is no physically interesting type of instability if $0 < n \leq 1$. But if $n > 3$, the polytropes are unstable if S_0 is the outer boundary beyond which there is a vacuum, a result known from the virial theorem. However, if S_0 is a boundary beyond which there is some other material, the incomplete polytrope within S_0 is stable only if its radius is less than a certain critical radius. This radius is calculated for five values of n in the range 4 to 6.

G. C. McVittie (Urbana, Ill.)

6317:

Jardetzky, Wenceslas S. Theories of figures of celestial bodies. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1958. xi+186 pp. \$6.50.

This brief treatise is a model of what a survey of a highly mathematical subject such as gravitational dynamics should be. No attempt is made to provide all the background knowledge of mathematics that may be required: if, for example, Lamé functions are to be used the reader is expected to look elsewhere for their detailed properties. The result is that the reader is guided through the maze of methods with a sure hand. The path leads from the theory of Maclaurin and Jacobi ellipsoids for liquid rotating masses, through the methods of Poincaré and the equilibrium figures of masses that are nearly ellipsoids, the ovoidal figures and stability considerations. Then comes an account of Liapounov's work on the correct method for the series expansion of the gravitational potential for a liquid mass in equilibrium, followed by an account of the improvements made by Lichtenstein, Wavre and others.

The second part of the book deals with zonal rotation problems and the question of convection, with some applications to the figure of the earth. Small oscillations are also treated and two chapters are devoted to the theory of systems that are partly liquid and partly solid. The last chapter, on the figures of compressible masses, should be of special interest to astronomers. It is shown why it is not safe to take results proved for liquid masses,

whether in rotational or non-rotational equilibrium, and apply them without examination to gaseous masses such as stars.

G. C. McVittie (Urbana, Ill.)

GEOPHYSICS

See also 6154.

6318:

Rossiter, J. R. On the application of relaxation methods to oceanic tides. Proc. Roy. Soc. London. Ser. A 248 (1958), 482-498.

Problems of tidal distribution in canals and oceans involve the solution of ordinary or partial differential equations of elliptic type. Both types are treated by relaxation methods, with emphasis on the choice of suitable independent variables which guarantee reasonable convergence of the numerical method, particularly in regions remote from the equator. The numerical result for a meridional canal compares well with an analytical solution, giving confidence in the results obtained for an ocean resembling the Atlantic.

L. Fox (Oxford)

6319:

Hantush, Madhi S. Non-steady flow to a well partially penetrating an infinite leaky aquifer. Proc. Iraqi Sci. Soc. 1 (1957), 10-19. (Arabic summary)

The author considers a well partially penetrating a leaky aquifer which is infinite in any horizontal direction and is bounded above and below by semi-pervious confining beds. The problem is to determine the drawdown s induced by this steady well under the assumption that the initial head is uniform. This leads to consideration of the boundary value problem: (1) $\nabla^2 s - s/B^2 = (\alpha - 1)\partial s/\partial t$ (B and α are constants), (2) $s(r, z, t) = 0$ at $t = 0$ and $r = \infty$, (3) $\partial s/\partial z = 0$ at $z = 0$ and $z = b$, (4) $\lim_{r \rightarrow 0} r f(\partial s/\partial r) dz = \text{const}$. A function satisfying (1) and (2) is obtained by use of the Laplace transform, and the vanishing of the flux at $z = 0$ and $z = b$ is obtained by the method of images. Both steady state and time dependent solutions are given in series forms and tables, and graphs show the dependence of s on distance and time for a typical set of values of the parameters B , b and α .

C. G. Maple (Ames, Iowa)

OPERATIONS RESEARCH AND ECONOMETRICS

See also 6161, 6162.

6320:

Morishima, Michio. Prices, interest and profits in a dynamic Leontief system. Econometrica 26 (1958), 358-380.

This article is one of many recent contributions to the mathematical theory of general economic equilibrium. It presents a series of interesting theorems, but their content and meaning may be difficult to understand, at least for readers who are not intimately familiar with the literature in this particular field of economic theory.

T. Haavelmo (Oslo)

6321:

Sisson, Roger L. Methods of sequencing in job shops — a review. Operations Res. 7 (1959), 10-29.

6322:

★Metzger, Robert W. *Elementary mathematical programming*. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1958. ix+246 pp. \$5.95.

According to the dust jacket, this book is "designed for the reader with a limited background in mathematics who wishes to understand the basic techniques and applications of mathematical programming." Except that the word "use" would be a bit more accurate than the word "understand", this is a good enough description. The most valuable aspects of the book are the very clear step-by-step explanations of the arithmetic of (1) solving the transportation problem of linear programming and (2) applying the simplex method and some variants. There are some ingenious case studies illustrating how to formulate industrial problems as linear programming problems, though the reader is not always adequately warned of complications likely to arise in practice. The emphasis is entirely on applications; no theorems are proved or even carefully stated, and with very scarce exceptions arithmetic is the only form of mathematics employed. In spite of the title, the treatment is limited to linear programming methods.

R. Dorfman (Cambridge, Mass.)

6323:

Croes, G. A. *A method for solving traveling-salesman problems*. *Operations Res.* 6 (1958), 791-812.

This paper does not "solve" the traveling-salesman problem, nor does it present any new or especially useful mathematical results. The paper does systematize, and extend slightly, one familiar technique for computing solutions of actual numerical cases. This technique makes use of the familiar facts that: a) the solution is invariant under addition of a constant to a line of the cost matrix; b) the solution tour has no "intersections"; and c) a trial solution composed entirely of zero costs can be improved upon only by introducing one or more negative costs. The author reports success in solving various numerical examples, including the 42-city problem of Dantzig, Fulkerson and Johnson [*J. Operations Res. Soc. Amer.* 2 (1954), 393-410; MR 17, 58], and illustrates his method by a 20-city symmetrical problem.

M. M. Flood (Ann Arbor, Mich.)

INFORMATION AND COMMUNICATION THEORY

See also 6150, 6157.

6324:

★Votavová, Libuše. *Ein Satz von Extremen der Entropie*. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 293-295. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

The author derives the well-known result that the maximum of $H = -\sum_{i=1}^n \mu_i \log \mu_i$ subject to $\mu_i \geq 0$, $\sum_{i=1}^n \mu_i = 1$, $j < n$, μ_1, \dots, μ_j fixed is $-\sum_{i=1}^j \mu_i \log \mu_i - c \log(c/n-j)$, where $c = 1 - \sum_{i=1}^j \mu_i$. He also shows that under the additional constraint $\mu_j + i \leq \mu_k$, $i = 1, \dots, n-j$, the minimum of H , if it exists, is $\sum_{i=1}^j \mu_i \log \mu_i - n \mu_k \log \mu_k - \mu \log \mu$, where $\mu_k = \max_{1 \leq i \leq j} \{\mu_i\}$, $M = [c/\mu_k]$ and $\mu = c - n \mu_k$.

R. A. Leibler (Wheaton, Md.)

6325:

★Pérez, Albert. *Notions généralisées d'incertitude, d'entropie et d'information du point de vue de la théorie de martingales*. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 183-208. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

Following the work of Kullback and Leibler [*Ann. Math. Statist.* 22 (1951), 79-86; MR 12, 623] the author develops certain results extending the notions of uncertainty, entropy, and information in the case of abstract probability spaces. In particular, the author develops these ideas in relation to increasing sequences of sub- σ -algebras of the probability space, thereby making it possible to apply results from the theory of martingales and semi-martingales as developed by Doob [*Stochastic processes*, Wiley, New York, 1953; MR 15, 445]. Let (X, \mathfrak{X}) be a measurable space. If $\mu \ll \lambda(\mathfrak{X})$ and $f(x)$ is a version of the corresponding Radon-Nikodym density then $-\log f(x)$ is defined as the generalized uncertainty of μ with respect to λ at the point $x \in (X, \mathfrak{X})$. The author's theorem 3 on the almost sure convergence of generalized uncertainties is: Let $\{\mathfrak{X}_n\}$ be an increasing sequence of sub- σ -algebras of \mathfrak{X} and let \mathfrak{X}_∞ be the σ -algebra generated by the algebra $\bigcup_{n=1}^\infty \mathfrak{X}_n$. If $\mu \ll \lambda(\mathfrak{X}_\infty)$ and $\{f_n(x), \mathfrak{X}_n, 1 \leq n \leq \infty\}$ is the λ -martingale of respective densities with $\int \log f_\infty(x) d\mu$ finite, then the sequence of generalized uncertainties $\{-\log f_n(x)\}$ converges almost surely $[\mu]$ to the generalized uncertainty $-\log f_\infty(x)$. If $\mu \ll \lambda(\mathfrak{X})$ and $f(x)$ the corresponding density with $\int \log f(x) d\mu$ finite, the preceding sequence of uncertainties converges to $-\log f(x)$ almost surely $[\mu]$ if and only if a version of the density $f(x)$ is measurable with respect to the σ -algebra \mathfrak{X}_∞ , that is, if \mathfrak{X}_∞ is a sufficient σ -algebra for the system of measures $\{\mu, \lambda\}$. The author defines the generalized entropy of μ with respect to λ , relative to the σ -algebra \mathfrak{X} , as the mean value with respect to μ of the generalized uncertainty, that is,

$$H_\lambda(\mu, \mathfrak{X}) = - \int \log f(x) d\mu = - \int f(x) \log f(x) d\lambda.$$

The author's theorem 7 on the generalized entropy, using the notation defined for theorem 3, is: If $\lim_{n \rightarrow \infty} H_\lambda(\mu, \mathfrak{X}_n)$ is finite then (i) a necessary and sufficient condition that the sequence of generalized uncertainties $\{-\log f_n\}$ converges in μ -mean is that $\mu \ll \lambda(\mathfrak{X}_\infty)$, (ii) if this last condition is satisfied then $\lim_{n \rightarrow \infty} H_\lambda(\mu, \mathfrak{X}_n) = H_\lambda(\mu, \mathfrak{X}_\infty)$, (iii) if $\mu \ll \lambda(\mathfrak{X})$ and $H_\lambda(\mu, \mathfrak{X})$ is finite, a necessary and sufficient condition that $\lim_{n \rightarrow \infty} H_\lambda(\mu, \mathfrak{X}_n) = H_\lambda(\mu, \mathfrak{X})$ is that \mathfrak{X}_∞ be a sufficient σ -algebra for the system of measures $\{\mu, \lambda\}$ on \mathfrak{X} . The author also derives some results on the discrimination between the probability measures μ and λ in terms of the generalized entropy. These include as a special case a limit theorem by McMillan [*Ann. Math. Statist.* 24 (1953), 196-219; MR 14, 1101]. Let $(X \times Y, \mathfrak{X} \times \mathfrak{Y}, \omega)$ be a probability space where the measurable space is the cartesian product of two measurable spaces (X, \mathfrak{X}) and (Y, \mathfrak{Y}) . Let μ and ν be the marginal probability measures induced by ω on \mathfrak{X} and \mathfrak{Y} respectively. Suppose that $\omega \ll \mu \times \nu(\mathfrak{X} \times \mathfrak{Y})$ and $f(x, y)$ is the corresponding Radon-Nikodym density. The generalized information corresponding to the probability space $(X \times Y, \mathfrak{X} \times \mathfrak{Y}, \omega)$ is defined as $I(\omega, \mathfrak{X} \times \mathfrak{Y}) = \int \log f(x, y) d\omega$. The author relates this information value to the difference between a marginal entropy and a mean conditional entropy and

derives a number of theorems similar to those on the generalized entropy. The author remarks that information can serve as a rational basis for a universal measure of the degree of stochastic dependence between two random entities since $I(\omega, \mathcal{X} \times \mathcal{Y})$ is zero if and only if the random entities considered are independent.

S. Kullback (Washington, D.C.)

6326:

★Pérez, Albert. Sur la théorie de l'information dans le cas d'un alphabet abstrait. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 209-243. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

This is essentially a continuation of the author's work reviewed above with the aim of developing the general theory of information in which the alphabet may be any abstract measurable space. The author discusses a probabilistic notion of discrimination, and relates this to an increasing sequence of σ -algebras in the sample space and the generalized notions of uncertainty, entropy, and information. In particular the author derives a generalized version of the theorem on the E (equipartition) property by McMillan [Ann. Math. Statist. 24 (1953), 196-219; MR 14, 1101] and a generalized version of the fundamental lemma of Feinstein [Res. Lab. Electron. Mass. Inst. Tech., Tech. Rep. no. 282 (1954); MR 17, 1098], the generalization residing in the fact that the results previously found for the case of finite alphabets are valid for any alphabets when the capacity of a channel is defined in terms of the generalized notions of information and rate of transmission of information. The author considers extensions of the notion of transmissibility and derives results which permit of the ordering of sources of information considered as functions of their information rates with regard to their transmissibility through a communication channel. In particular, the theorem designated by Khinchin as the first Shannon theorem [Shannon, Bell System Tech. J. 27 (1948), 379-423, 623-656; MR 10, 133; Hincin, Uspehi Mat. Nauk (N.S.) 11 (1956), no. 1 (67), 17-75; MR 17, 1098; A. I. Khinchin (Hincin), Mathematical foundations of information theory; translated by R. A. Silverman and M. D. Friedman, Dover Publ., New York, N.Y., 1957; MR 19, 1148] is derived as a special case of the general results.

S. Kullback (Washington, D.C.)

6327:

★Pérez, Albert. Sur la convergence des incertitudes, entropies et informations échantillon (sample) vers leurs valeurs vraies. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 245-252. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

This is essentially a continuation of the author's work reviewed above [6325] to study the conditions for the convergence of sample uncertainties, entropies, and informations to their true values. The sample values of uncertainty, entropy and information are defined and suitable conditions derived. Space does not permit presenting the detailed definitions here.

S. Kullback (Washington, D.C.)

CONTROL SYSTEMS

See also 5736.

6328:

Širokorad, B. V. On existence of a cycle beyond absolute stability conditions of a three-dimensional system. Avtomatika i Telemekhanika 19 (1958), 953-967. (Russian. English summary)

The system of differential equations of the form

$$\dot{x}_1 = -\rho_1 x_1 + f(\sigma), \quad \dot{x}_2 = -\rho_2 x_2 + f(\sigma),$$

$$\dot{\sigma} = \beta_1 x_1 + \beta_2 x_2 - f(\sigma)$$

is studied in the critical case $\rho_1 = 0$ and with $-\beta_1 = \beta_2 > \rho_2 > 0$. Sufficient conditions are given for the function $f(\sigma)$ to insure that the system is ultimately bounded [i.e., there is an $R > 0$ such that for each solution $x_1^2(t) + x_2^2(t) + \sigma^2(t) < R^2$ for all t sufficiently large] and that there exists a non-trivial periodic solution. Feedback systems give rise to such differential equations, and the author gives two examples: a flight control system and a triode oscillation.

J. P. LaSalle (Baltimore, Md.)

HISTORY AND BIOGRAPHY

See also 6054.

6329:

Neugebauer, O. Ptolemy's Geography, book VII, chapters 6 and 7. Isis 50 (1959), 22-29.

A translation into English of the indicated chapters, followed by a mathematical commentary. The question under discussion is the representation of the terrestrial globe in the plane as seen by an observer who is placed in the plane of the parallel of latitude which passes through Syene.

6330:

Thorndike, Lynn. Notes upon some medieval astronomical, astrological and mathematical manuscripts at Florence, Milan, Bologna and Venice. Isis 50 (1959), 33-50.

6331:

Rychlík, Karel. Betrachtungen aus der Logik in Bolzanos handschriftlichem Nachlasse. Czechoslovak Math. J. 8(83) (1958), 197-202. (1 plate) (Russian summary)

Der Verfasser bringt in diesem Aufsatz den Inhalt von Bolzanos Arbeit "Von der mathematischen Lehrart" und den § 8 dieser Arbeit, einen Auszug, der für die mathematische Logik von besonderer Bedeutung ist.

Zusammenfassung des Autors

6332:

Lebesgue, Henri. Notices d'histoire des mathématiques. Avec une introduction de L. Félix. Monographies de L'Enseignement Mathématique, no. 4. L'Enseignement mathématique, Geneva, 1958. 116 pp. (1 plate) 20 francs suisses.

The book contains a photograph of Lebesgue and an introduction by L. Félix, followed by 7 sections with the titles: Commentaires sur l'oeuvre de F. Viète, l'oeuvre mathématique de Vandermonde, la vie et les travaux de

Camille Jordan, notice sur René-Louis Baire, un travail mathématique de André-Marie Ampère, les professeurs de mathématiques du Collège de France: Humbert et Jordan, Roberval et Ramus; extraits de la correspondance de H. Lebesgue.

6333:

Ščerban', O. N. Growth of science in Soviet Ukraine. Dopovidi Akad. Nauk Ukraïn. RSR 1957, 423-430. (Ukrainian)

6334:

Anonymous. Development of physico-mathematical sciences in Armenia in the period of Soviet rule. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk 10 (1957), no. 5, 3-18. (Russian)

A discussion of Armenian activity from 1921 to the present, under the headings: mathematics, mechanics, astrophysics, physics.

6335:

Lapko, A. F.; and Lyusternik, L. A. Mathematical sessions and conferences in the USSR. Uspehi Mat. Nauk 13 (1958), no. 5(83), 121-166. (Russian)

Complementary material to the authors' earlier article [same Uspehi 12 (1957), no. 6(78), 47-130; MR 19, 1029].

6336:

Anonymous. Obituary: Vasilii Zaharovič Vlasov. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1958, no. 10, 3-4. (1 plate) (Russian)

A short biography with one photograph. There is no bibliography.

6337:

Kantorovič, L. V.; and Natanson, I. P. Grigorii Mihalovič Fihhtengol'c (on his seventieth birthday). Vestnik Leningrad. Univ. 13 (1958), no. 7, 5-13. (1 plate) (Russian)

A general and scientific biography, with one photograph and a bibliography of 39 entries.

6338:

★Fubini, Guido. Opere scelte. Vol. II. A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. 350 pp. 3500 Lire.

For Vol. I (1957) see MR 19, 827. The present volume contains 29 articles, from 1904 through 1910, on differential geometry, partial differential equations, calculus of variations, integral equations.

6339:

★Peano, Giuseppe. Opere scelte. Vol. II. Logica matematica, Interlingua ed Algebra della grammatica. A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vi+518 pp. 5000 Lire.

For Vol. I (1957) see MR 19, 827. The present volume contains the 29 articles promised in the preface of Vol. I.

6340:

★Ramanujan, Srinivasa. Notebooks. Vols. 1, 2. Tata Institute of Fundamental Research, Bombay, 1957. Vol. 1. vi+351 pp.; Vol. 2. vi+393 pp. Two vols. Rs. 100.

The mathematical world will be grateful to K. Chandra-

sekharan for the publication of Ramanujan's famous notebooks. It is unfortunate, however, that the notebooks are little more than a record of the results obtained by Ramanujan (in the theory of infinite series, definite integrals, continued fractions, "modular equations", etc.) and not of his methods of obtaining these results. To readers of the well-known publications *Collected Papers of S. Ramanujan* [University Press, Cambridge, 1927; we refer to this as R.C.P.] and Hardy's *Ramanujan* [University Press, Cambridge, 1940; MR 3, 71; we refer to this as R], the brilliant formulae of Ramanujan, of which these notebooks are so full, are no surprise. We quote a few at random: (1) If ϕ and f are continued fractions of the form

$$\phi = \frac{x}{1 + \frac{x^5}{1 + \frac{x^{10}}{\dots}}}, \quad f = \frac{x^{1/5}}{1 + \frac{x}{1 + \frac{x^2}{\dots}}},$$

then

$$f^5 = \phi \frac{1 - 2\phi + 4\phi^2 - 3\phi^3 + \phi^4}{1 + 3\phi + 4\phi^2 + 2\phi^3 + \phi^4}$$

[Notebooks, vol 1, p. 344; R., p. 8; R.C.P., p. 27].

$$(2) \quad 1 + 480 \sum_{n=1}^{\infty} \sigma_1(n) x^n = \{1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) x^n\}^2;$$

here, $\sigma_a(n) = \sum_{d|n} d^a$ [Notebooks, vol. 1, p. 256; R.C.P., p. 141].

$$(3) \quad \text{If } (1 - e^{-\pi\sqrt{n}})(1 - e^{-3\pi\sqrt{n}})(1 - e^{-5\pi\sqrt{n}}) \dots = (2)^{1/4} e^{-\pi\sqrt{n}/24} g_n$$

then $g_{58} = (\frac{1}{2}(5 + \sqrt{29}))^{\frac{1}{2}}$ [Notebooks, vol. 1, p. 318; the expressions for g_{522} and g_{630} on the same page are too complicated to be quoted here].

$$(4) \quad \frac{16}{\pi} = 5 + \frac{47}{64} \left(\frac{1}{2}\right)^3 + \frac{89}{64^2} \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^3 + \frac{131}{64^3} \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^3 + \dots$$

[Notebooks, vol. 2, p. 355; R.C.P., p. 36; similar results in R., p. 7, (1.2), (1.3) and (1.4)].

One cannot do justice to the notebooks in such a short review; the reader will have to look at the original to satisfy his curiosity. We shall content ourselves by listing by title only a few of the topics treated in the notebooks: magic squares, number of primes not exceeding x , Bernoulli's numbers (Ramanujan rediscovered von Staudt's theorem), asymptotic expansions of integrals and of functions defined by infinite series, numbers of the form $x^3 \pm y^3$ up to 1000.

One misses, e.g., any scribbles on Ramanujan's unproved conjecture: $|\tau(n)| \leq n^{11/2} \sigma_0(n)$, where $\sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}$. S. Chowla (Boulder, Colo.)

GENERAL

See also 5929.

6341:

★Pidek-Lopuszańska, H.; Słobodziński, W.; and Urbanik, K. Matematyka dla chemików. [Mathematics for chemists.] Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 475 pp.

This book is written for first and second year students of chemistry at Polish universities and technical schools. The contents are: numbers and vectors; plane and solid analytic geometry; differential and integral calculus

(corresponding to the usual first course, augmented by some topics of advanced calculus); and several brief chapters introducing determinants and matrices, Fourier series, calculus of variations, group theory, ordinary and partial differential equations etc.

There are some problems, an index and a little historical appendix (in which the mathematician Joseph Fourier is misidentified with the 'utopian socialist' Charles).

Ž. A. Melzak (Montreal, P. Q.)

6342:

***Bayley, F. J.** *An introduction to fluid dynamics.* Interscience Publishers Inc., New York, 1958. viii+215 pp. (3 plates) \$4.50.

This work represents a college text-book suitable for the upper years of an undergraduate course.

6343:

***Maxwell, E. A.** *Fallacies in mathematics.* Cambridge University Press, New York, 1959. 95 pp. \$2.95.

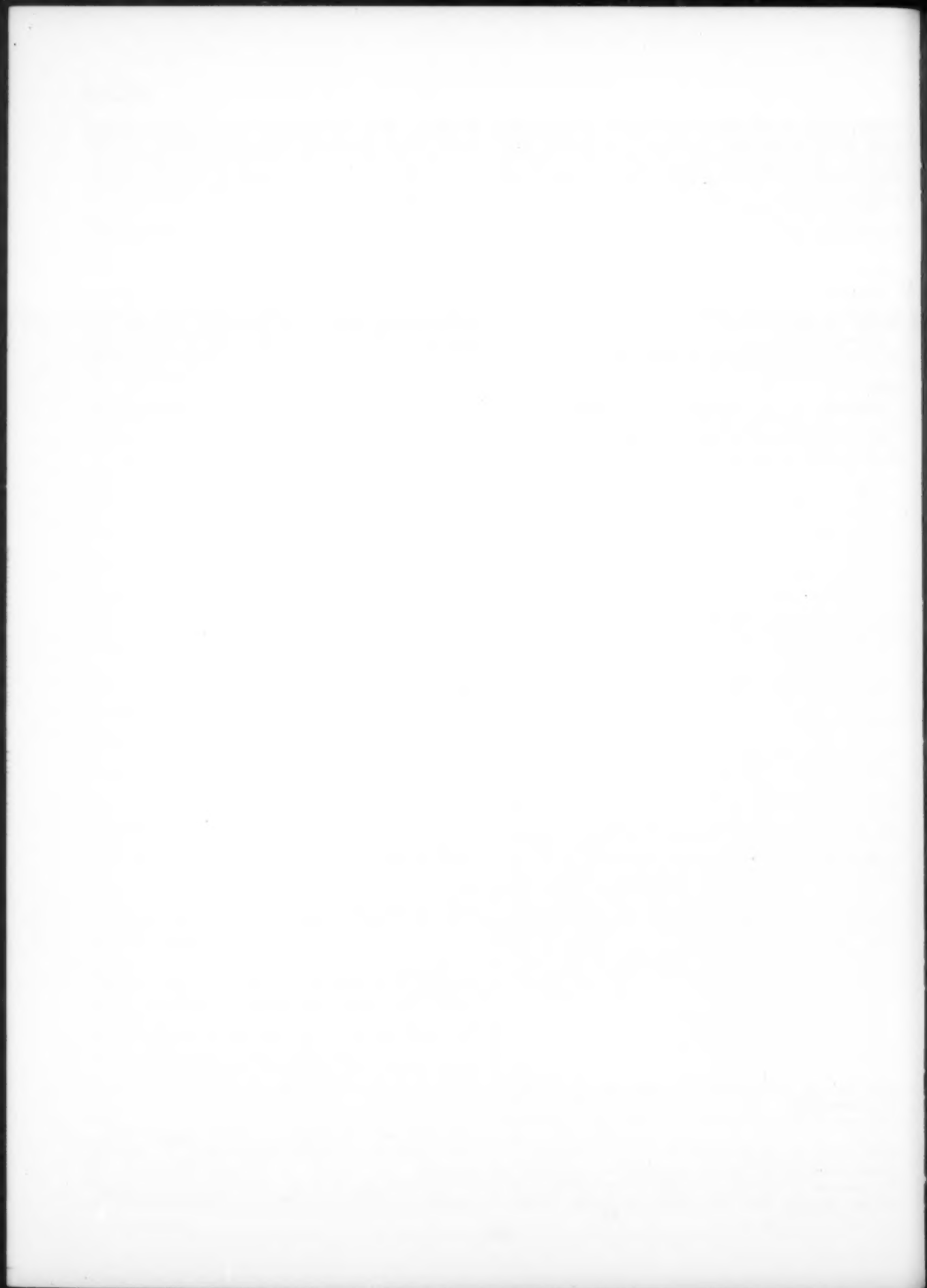
Many of the fallacies in this book will be interesting to American high school students; some require a knowledge

of calculus. There are 11 chapters: Mistake, howler and fallacy; Four geometrical fallacies; Digression on elementary geometry; Analysis of the fallacy of the isosceles triangle; Analysis of other geometrical fallacies; Fallacies in algebra and trigonometry; In differentiation; In integration; Fallacies based on the circular points at infinity; Some 'limit' fallacies; Some miscellaneous howlers.

6344:

***Słownik polsko-rosyjsko-angielski statystyki matematycznej i statystycznej kontroli jakości produkcji.** [Polish-Russian-English dictionary of mathematical statistics and statistical quality control of production.] Polska Akademia Nauk, Instytut Matematyczny: Zastosowania Matematyki. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 48 pp. zł. 7.

A list of 540 terms arranged in the alphabetical order of the Polish words. The terms are numbered consecutively and the list is followed by a Russian and an English index with references to the corresponding numbers.



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